METASTABILITY OF THE MEISSNER STATE IN THE THREE-DIMENSIONAL GINZBURG–LANDAU MODEL WITH PINNING

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Abstract. We study the persistence of the Meissner state in the three-dimensional magnetic Ginzburg–Landau model with a spatially dependent pinning term. Factoring out the equilibrium profile associated with the pinning term reduces the problem to the analysis of a weighted Ginzburg–Landau functional. In the extreme type-II regime, we construct locally minimizing vortexless configurations for applied fields extending far beyond the first critical field. We obtain quantitative estimates showing that these configurations remain close, up to gauge transformations, to a Meissner-type state determined by a weighted London problem. We also prove the uniqueness, up to gauge transformations, of the locally minimizing Meissner-type branch within a quantitatively controlled class of local minimizers. A central difficulty is to derive elliptic estimates that remain uniform in the Ginzburg–Landau parameter in the presence of the resulting spatially dependent weight. Under a small-BMO assumption on this weight, we obtain the required regularity and extend to the three-dimensional pinned setting the metastability picture previously established for the homogeneous model.

1. Introduction

The magnetic Ginzburg–Landau model [GL50] provides a phenomenological description of superconducting materials under applied magnetic fields. Its configurations depend on a complex-valued order parameter, which describes the local superconducting state, and on a magnetic vector potential for the induced magnetic field. A fundamental parameter of the model is the Ginzburg–Landau parameter κ, which distinguishes between type-I and type-II superconductors. Throughout this work, we write $\varepsilon = \kappa^{-1}$ and consider the regime $\varepsilon \to 0$, corresponding to extreme type-II superconductors.

At low applied fields, a superconducting sample is expected to be in the Meissner state, characterized by the expulsion of the magnetic field from the interior of the material. As the intensity h_{ex} of the applied field increases, this state eventually ceases to be globally minimizing and configurations containing vortices become energetically favorable. Vortices are defects in the superconducting state that carry quantized magnetic flux; in three dimensions, they concentrate along curves and are therefore referred to as vortex lines or filaments. The

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Date: October 30, 2026.

2020 *Mathematics Subject Classification.* Primary 35Q56; Secondary 35J20, 35B40, 82D55.

Key words and phrases. Ginzburg–Landau model, superconductivity, pinning, Meissner state, metastability, local minimizers, elliptic regularity.

Funding information: ANID FONDECYT 1231593.

threshold at which they first become energetically favorable is known as the first critical field and is denoted by H_{c_1}. In the extreme type-II regime, its leading order is proportional to $|\log \varepsilon|$.

The first critical field is, however, an energetic threshold and does not necessarily coincide with the loss of stability of the Meissner state. Although vortex configurations have lower energy above H_{c_1}, a locally minimizing vortexless solution may persist for substantially larger applied fields. The field at which this branch loses stability is known as the *superheating field*, denoted by H_{sh}.

This distinction was made rigorous in the three-dimensional homogeneous setting in [Rom19a]. More precisely, it was proved that below the first critical field the Meissner solution is, up to gauge transformations, the unique global minimizer of the Ginzburg–Landau energy. Beyond this threshold, a Meissner-type solution persists as a local minimizer at least for $h_{\mathrm{ex}} = o(\varepsilon^{-\frac{1}{3}})$, and is unique, up to gauge transformations, at least for $h_{\mathrm{ex}} = o(\varepsilon^{-\frac{1}{4}})$. Thus, when the applied field is gradually increased, vortex nucleation is expected to be delayed beyond H_{c_1} and to occur only when the vortexless branch becomes unstable. This loss of stability is associated with the superheating field, which is expected to be of order ε^{-1}; see, for instance, [Cha95].

The purpose of the present work is to investigate to what extent this picture remains valid for the three-dimensional Ginzburg–Landau functional in the presence of material inhomogeneities described by a spatially dependent pinning term.

1.1. **The Ginzburg–Landau model with pinning.** The performance of type-II superconductors is strongly affected by the motion of vortices, which produces dissipation and limits the current that the material can carry while remaining superconducting. Material inhomogeneities may restrict this motion by acting as pinning sites for vortices. This provides one of the main motivations for introducing spatially dependent coefficients into the Ginzburg–Landau model. Such an approach goes back to Likharev [Lik79], who proposed to account for spatial variations of the superconducting properties by allowing the coefficients in the energy functional to depend on the position.

Vortex pinning has been the subject of considerable interest in the physics of superconductivity. The review [KWG+16], for instance, discusses how different types of defects and inclusions give rise to complex pinning landscapes, which influence vortex configurations and play an important role in enhancing the critical current, namely the maximal current that can be carried without losing the superconducting behavior. In a more microscopic direction, experimental work on FeSe-based superconductors has provided direct evidence that local defects can act as pinning centers for vortices [CLC+24]. These works provide physical motivation for considering spatially dependent pinning terms in the Ginzburg–Landau functional rather than restricting the analysis to the classical homogeneous model. For a more detailed discussion of the physics underlying vortex pinning and its implications, see, for instance, [Lik79, Tin96, CR97] and the references therein.

After nondimensionalization of the physical constants, the Ginzburg–Landau functional with pinning, which models an inhomogeneous superconducting sample in an applied magnetic field at a fixed temperature below the critical one, is given by

$$GL_\varepsilon(\mathsf{u},\mathsf{A}) = \frac{1}{2}\int_\Omega \left(|\nabla_{\mathsf{A}}\mathsf{u}|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon(x)-|\mathsf{u}|^2)^2\right) + \frac{1}{2}\int_{\mathbb{R}^3}|\mathsf{H}-H_{\mathrm{ex}}|^2. \tag{1.1}$$

Here, Ω is a bounded domain of $\mathbb{R}^3$, which we assume to be simply connected with C^2 boundary. The function $\mathsf{u}:\Omega\to\mathbb{C}$ is the *order parameter*; its squared modulus, which represents the relative density of Cooper pairs of superconducting electrons in the microscopic BCS theory [BCS57], indicates the local state of the superconductor. The vector field $\mathsf{A}:\mathbb{R}^3\to\mathbb{R}^3$ is the electromagnetic vector potential of the induced magnetic field $\mathsf{H}=\operatorname{curl}\mathsf{A}$, and ∇_{A} denotes the covariant gradient $\nabla - i\mathsf{A}$. The external, or applied, magnetic field is assumed to be of the form $H_{\mathrm{ex}} = h_{\mathrm{ex}}H_{0,\mathrm{ex}}$, where $H_{0,\mathrm{ex}}\in L^\infty(\mathbb{R}^3,\mathbb{R}^3)$ is a fixed divergence-free vector field and $h_{\mathrm{ex}}\geq 0$ is a tunable parameter representing the intensity of the external field.

Finally, a_ε is a function that accounts for inhomogeneities in the material. We assume that $a_\varepsilon\in L^\infty(\Omega)$ and that it takes values in $[b,1]$, where $b\in(0,1)$ is a constant independent of ε. Regions where $a_\varepsilon = 1$ correspond to sites without inhomogeneities, whereas regions where $a_\varepsilon<1$ may act as pinning sites.

The Ginzburg–Landau model is a $\mathbb{U}(1)$-gauge theory, in which all the physically relevant quantities are invariant under the gauge transformations

$$\mathsf{u}\to\mathsf{u}e^{i\Phi},\qquad \mathsf{A}\to\mathsf{A}+\nabla\Phi,$$

where Φ is any sufficiently regular real-valued function. Two important gauge-invariant quantities are the supercurrent and the vorticity, respectively defined, for any sufficiently regular configuration (u,A), by

$$j(\mathsf{u},\mathsf{A}) = (i\mathsf{u},\nabla_{\mathsf{A}}\mathsf{u}),\qquad \mu(\mathsf{u},\mathsf{A}) = \operatorname{curl} j(\mathsf{u},\mathsf{A}) + \operatorname{curl}\mathsf{A},$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$.

Another gauge-invariant quantity that will play an important role throughout this paper is the free energy restricted to Ω, defined by

$$F_{\varepsilon,\Omega}(u,A) = \frac{1}{2}\int_\Omega\left(|\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 + |\operatorname{curl} A|^2\right).$$

1.2. **A brief historical overview.** The homogeneous Ginzburg–Landau model has a long history of mathematical analysis. The two-dimensional theory was developed in a series of important works; we refer, in particular, to the foundational monographs [BBH94, SS07] and the references therein for comprehensive accounts of the subject.

In the two-dimensional homogeneous setting, an important step towards the rigorous understanding of the superheating field was made by Serfaty in [Ser99]. There, a branch of vortexless local minimizers was constructed and shown to be unique, up to gauge transformations, in an explicit regime of applied fields extending far beyond the first critical field.

The three-dimensional homogeneous problem is considerably more involved. In two dimensions, vortices are localized near isolated points and can be described in terms of their topological degrees. In three dimensions, they concentrate instead along curves, and their analysis requires tools from geometric measure theory. Among the works that laid the foundations of

the three-dimensional theory, we mention [Riv95, LR99, San01, BJOS12, BJOS13], where the limiting behavior of vortex filaments and the asymptotic structure of the Ginzburg–Landau energy were investigated in different regimes. More refined tools allowing one to describe vortex filaments at the ε-level were later developed in [Rom19b]; see also [Rom19a, RSS23, RSS25] for applications to the behavior of minimizers near the first critical field.

These ε-level estimates were a key ingredient in extending the approach of Serfaty [Ser99] to the three-dimensional homogeneous setting in [Rom19a]. This extension established the persistence and uniqueness of the Meissner branch in suitable regimes of applied fields, providing the first three-dimensional counterpart of the two-dimensional theory. More recently, the approach was adapted to the two-dimensional Ginzburg–Landau model with pinning in [DVR26b].

The Ginzburg–Landau model with pinning has been extensively studied in two dimensions, in connection with a wide variety of questions concerning vortex nucleation, localization, and interaction. We refer to [DVR26b] and the references therein for an overview of this literature. By contrast, its three-dimensional counterpart has received comparatively little mathematical attention. In [DS12], a non-magnetic problem with pinning and nontrivial Dirichlet boundary conditions was considered, and the limiting singular set was shown to form a geodesic link determined by the pinning landscape. Although most of their results concern the two-dimensional setting, Dos Santos, Rodiac, and Sandier [DSRS23] also considered the full Ginzburg–Landau energy in three dimensions with rapidly oscillating pinning terms. Assuming that the average of the pinning term in the periodic case, or its expectation in the random stationary case, is equal to 1, they proved that, as $\varepsilon \to 0$, the limiting functional coincides with the one obtained in the homogeneous case by Baldo, Jerrard, Orlandi, and Soner [BJOS12].

More recently, the development of a weighted counterpart of the ε-level estimates introduced in [Rom19b] was instrumental in the study of the first critical field for the three-dimensional magnetic Ginzburg–Landau functional with pinning in [DVR26a, Rom26]. The tools developed in these works also provide a basis for addressing the different question considered here, concerning the persistence and uniqueness of the Meissner branch. They allow us to follow, in broad terms, the strategy used in the homogeneous setting, although its implementation in the presence of pinning is not direct. In particular, the construction and uniqueness arguments require elliptic estimates for equations whose coefficients depend on the pinning profile. Both the gain in integrability and the constants involved must be controlled uniformly in ε, which requires a careful choice of gauges and suitable control of the oscillations of the weight.

1.3. **Main results.** Throughout the paper, $C > 0$ denotes a constant independent of ε and h_{ex}, whose value may change from line to line and may depend on the fixed parameters of the problem.

Since $\operatorname{div} H_{0,\mathrm{ex}} = 0$ in $\mathbb{R}^3$, in accordance with the nonexistence of magnetic monopoles in Maxwell's theory of electromagnetism, there exists a vector potential $A_{0,\mathrm{ex}} \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\operatorname{curl} A_{0,\mathrm{ex}} = H_{0,\mathrm{ex}} \quad \text{in } \mathbb{R}^3.$$

We define

$$A_{\mathrm{ex}} := h_{\mathrm{ex}} A_{0,\mathrm{ex}}.$$

Observe that we can choose $A_{0,\mathrm{ex}}$ in the Coulomb gauge in Ω, namely,

$$\begin{cases} \operatorname{div} A_{0,\mathrm{ex}} &= 0 \quad \text{in } \Omega \\ A_{0,\mathrm{ex}} \cdot \nu &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where hereafter ν denotes the outer unit normal to $\partial\Omega$. The natural space for the minimization of GL_ε in three dimensions is $H^1(\Omega,\mathbb{C}) \times [A_{\mathrm{ex}} + H_{\mathrm{curl}}]$, where

$$H_{\mathrm{curl}} := \left\{ A \in H^1_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{R}^3) \;\middle|\; \operatorname{curl} A \in L^2(\mathbb{R}^3,\mathbb{R}^3) \right\}.$$

Let us also introduce the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3,\mathbb{R}^3)$, defined as the completion of $C^\infty_0(\mathbb{R}^3,\mathbb{R}^3)$ with respect to the norm $\|\nabla(\,\cdot\,)\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$. We denote by

$$\dot{H}^1_{\mathrm{div}=0} := \left\{ A \in \dot{H}^1(\mathbb{R}^3,\mathbb{R}^3) \;\middle|\; \operatorname{div} A = 0 \text{ in } \mathbb{R}^3 \right\}$$

its divergence-free subspace. For every $A \in \dot{H}^1_{\mathrm{div}=0}$, we have

$$\|A\|_{\dot{H}^1_{\mathrm{div}=0}} := \|\nabla A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} = \|\operatorname{curl} A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}. \tag{1.2}$$

These spaces will be used below to formulate the convergence of the magnetic vector potential and to fix suitable gauges. Before stating our main results, we introduce a useful decomposition of the order parameter, whose underlying idea goes back to Lassoued and Mironescu [LM99], that isolates the effect of the pinning term in the energy. To this end, we first consider the scalar profile induced by the pinning term. More precisely, let ρ_ε be the unique positive real-valued function satisfying

$$\begin{cases} -\Delta\rho_\varepsilon &= \dfrac{\rho_\varepsilon}{\varepsilon^2}(a_\varepsilon - \rho_\varepsilon^2) \quad \text{in } \Omega \\ \dfrac{\partial\rho_\varepsilon}{\partial\nu} &= 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The function ρ_ε represents the equilibrium modulus selected by the first two terms of (1.1) in the absence of a magnetic field. Indeed, while the potential term favors the pointwise relation $|\mathsf{u}|^2 = a_\varepsilon$, the gradient term penalizes the spatial variations required to reproduce a_ε. Equation (1.3) reflects the balance between these two competing effects, and ρ_ε^2 may therefore be interpreted as a regularized version of a_ε.

Since ρ_ε already incorporates the variations in the modulus of the order parameter induced by the pinning term, it is natural to factor out this profile by writing $\mathsf{u} = \rho_\varepsilon u$. The new order parameter u then describes the remaining variations relative to the preferred scalar profile ρ_ε. Letting

$$E_\varepsilon(u,A) := \frac{1}{2}\int_\Omega \left(|\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon - |u|^2)^2 \right), \qquad E_\varepsilon(u) = E_\varepsilon(u,0),$$

we have

$$E_\varepsilon(\rho_\varepsilon u, A) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \left(\rho_\varepsilon^2 |\nabla_A u|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2 \right). \tag{1.4}$$

Therefore, the analysis of (1.1) can be reduced to that of the weighted Ginzburg–Landau functional

$$\frac{1}{2}\int_\Omega \left(\rho_\varepsilon^2 |\nabla_{\mathsf{A}} u|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2 \right) + \frac{1}{2}\int_{\mathbb{R}^3} |\mathsf{H} - H_{\mathrm{ex}}|^2.$$

We refer to [DVR26a, Section 2] for details. The influence of the pinning term is thus incorporated into the spatially dependent weights ρ_ε^2 and ρ_ε^4. In particular, the estimates used in the construction and analysis of the Meissner solution, which we introduce next, must account for these weights and remain uniform as $\varepsilon \to 0$.

The Ginzburg–Landau model admits a distinguished configuration, unique modulo gauge transformations, that we refer to as the Meissner state, in reference to the Meissner effect in physics. It is obtained by minimizing the Ginzburg–Landau energy under the constraint $|\mathsf{u}| = \rho_\varepsilon$. More precisely, for each vector potential A, the phase ϕ_A is chosen to minimize

$$\frac{1}{2}\int_\Omega \rho_\varepsilon^2 |A - \nabla\phi|^2.$$

Equivalently, $A - \nabla\phi_A$ is in the weighted Coulomb gauge associated with the weight ρ_ε^2, namely,

$$\left\{ \begin{array}{rcl} \operatorname{div}(\rho_\varepsilon^2(A - \nabla\phi_A)) &=& 0 \quad \text{in } \Omega \\ (A - \nabla\phi_A)\cdot\nu &=& 0 \quad \text{on } \partial\Omega. \end{array} \right. \tag{1.5}$$

This reduces the constrained problem to the minimization of a functional depending only on the vector potential; we refer to [DVR26a, Section 2.3] for the precise construction and discuss the weighted Coulomb gauge in more detail in Section 2.1.

We denote by A_ε^0 the minimizer of this reduced functional, chosen so that

$$A_\varepsilon^0 - A_{0,\mathrm{ex}} \in \dot{H}^1_{\mathrm{div}=0}, \tag{1.6}$$

and let $\phi_\varepsilon^0 := \phi_{A_\varepsilon^0}$. The Meissner state then takes the form

$$(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, h_{\mathrm{ex}} A_\varepsilon^0),$$

where ϕ_ε^0 and A_ε^0 depend on the domain Ω, the normalized applied field $H_{0,\mathrm{ex}}$, and the weight function ρ_ε.

In view of (1.5), the Helmholtz–Hodge decomposition yields the existence of a unique vector field B_ε^0 such that

$$\rho_\varepsilon^2(A_\varepsilon^0 - \nabla\phi_\varepsilon^0) = \operatorname{curl} B_\varepsilon^0 \quad \text{in } \Omega, \tag{1.7}$$

with

$$\left\{ \begin{array}{rcl} \operatorname{div} B_\varepsilon^0 &=& 0 \quad \text{in } \Omega \\ B_\varepsilon^0 \times \nu &=& 0 \quad \text{on } \partial\Omega. \end{array} \right. \tag{1.8}$$

The estimates resulting directly from the minimization give, in particular,

$$\|A_\varepsilon^0 - A_{0,\mathrm{ex}}\|_{\dot{H}^1_{\mathrm{div}=0}} + \|\nabla\phi_\varepsilon^0\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{curl} B_\varepsilon^0\|_{L^2(\Omega,\mathbb{R}^3)} \le C.$$

The Euler–Lagrange equation satisfied by A_ε^0, which may be viewed as a weighted London equation for the vector potential associated with the induced magnetic field $\operatorname{curl} A_\varepsilon^0$, is

$$\operatorname{curl}^2(A_\varepsilon^0 - A_{0,\mathrm{ex}}) + \chi_\Omega \operatorname{curl} B_\varepsilon^0 = 0 \quad \text{in } \mathbb{R}^3.$$

From (1.6), we deduce that

$$-\Delta(A_\varepsilon^0 - A_{0,\mathrm{ex}}) = -\chi_\Omega \operatorname{curl} B_\varepsilon^0 \quad \text{in } \mathbb{R}^3.$$

Since $\chi_\Omega \operatorname{curl} B_\varepsilon^0 \in L^2(\mathbb{R}^3,\mathbb{R}^3)$, interior elliptic regularity, applied with a cut-off supported in a sufficiently large ball containing $\overline{\Omega}$, yields

$$\|A_\varepsilon^0 - A_{0,\mathrm{ex}}\|_{W^{2,2}(\Omega,\mathbb{R}^3)} \leq C.$$

By Sobolev embedding, we then obtain

$$\|A_\varepsilon^0 - A_{0,\mathrm{ex}}\|_{W^{1,6}(\Omega,\mathbb{R}^3)} \leq C.$$

On the other hand, since $A_{0,\mathrm{ex}}$ is in the Coulomb gauge in Ω, Lemma B gives

$$\|A_{0,\mathrm{ex}}\|_{W^{1,6}(\Omega,\mathbb{R}^3)} \leq C\|H_{0,\mathrm{ex}}\|_{L^6(\Omega,\mathbb{R}^3)}.$$

Consequently,

$$\|A_\varepsilon^0\|_{W^{1,6}(\Omega,\mathbb{R}^3)} \leq C.$$

By Sobolev embedding, we conclude that

$$\|A_\varepsilon^0\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C. \tag{1.9}$$

Since $b \leq \rho_\varepsilon^2 \leq 1$ (see Proposition 2.1), the classical result of Gröger [Grö89] (see also [Mey63] for Dirichlet boundary conditions) establishes that there exists $p_0 > 2$ such that

$$\|\nabla\phi_\varepsilon^0\|_{L^{p_0}(\Omega,\mathbb{R}^3)} \leq C\|A_\varepsilon^0\|_{L^{p_0}(\Omega,\mathbb{R}^3)}, \tag{1.10}$$

where both p_0 and C depend on b but not on ε. For our analysis, however, we need (1.10) to hold uniformly in ε for the exponents specified in the statements of our main results. This is one of the points where the presence of the spatially dependent weight creates substantial additional difficulties. In particular, both the gain in integrability and the constants in the relevant elliptic estimates must be controlled uniformly in ε.

In the course of writing this article, we realized that there is a mistake in the proof of [DVR26a, Proposition 2.2] concerning the regularity of $\nabla\phi_\varepsilon^0$. There, it was incorrectly concluded that (1.10) holds uniformly in ε for any $p_0 > 2$. In fact, extending this estimate beyond the exponent provided by the general theory requires either further assumptions on ρ_ε or an ε-dependent constant which, loosely speaking, depends on a suitable control of the oscillations of ρ_ε^2 at sufficiently small scales; see, for instance, [BW05].In a work in progress [DVRY], we will show that, under the main assumption on ρ_ε in [DVR26a], estimate (1.10) holds for any $p_0 > 2$, with a constant $C = C(\varepsilon) = |\log\varepsilon|^M$ for some $M > 0$. This estimate still suffices to establish the results of that paper.

To ensure that (1.10) holds for the required exponent, we fix $p_0 > 2$, as specified in the statements of our main results, and assume hereafter that there exist constants $r_0 > 0$ and $\delta_0 = \delta_0(b, p_0, r_0) > 0$ such that, for all sufficiently small ε,

$$[\rho_\varepsilon^2]_{BMO_{r_0}(\Omega)} \leq \frac{\delta_0}{2}. \tag{1.11}$$

Under this assumption, [BW05, Theorem 2] yields (1.10). Let us also remark that (1.11) is automatically satisfied whenever b is sufficiently close to 1.

In view of (1.7), combining (1.9) and (1.10), we also obtain

$$\|\operatorname{curl} B_\varepsilon^0\|_{L^{p_0}(\Omega,\mathbb{R}^3)} \leq C. \tag{1.12}$$

Moreover, in view of (1.8), [AS13, Corollary 3.4] yields

$$\|B_\varepsilon^0\|_{W^{1,p_0}(\Omega,\mathbb{R}^3)} \leq C. \tag{1.13}$$

We are now in a position to state our first main result. It establishes the existence of a locally minimizing vortexless configuration in a range of applied fields extending far beyond the first critical field. In addition to its local minimality, we obtain quantitative estimates showing that this configuration is close, up to gauge transformations, to the Meissner state introduced above.

Theorem 1.1. *Let $\alpha \in \left(0, \frac{1}{4}\right)$ and suppose that*

$$p_0 > \frac{3}{1-3\alpha}.$$

There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$, if $h_{\mathrm{ex}} \leq \varepsilon^{-\alpha}$, then there exists a vortexless configuration $(\mathsf{u}, \mathsf{A}) = (\rho_\varepsilon u e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, A + h_{\mathrm{ex}} A_\varepsilon^0)$ which locally minimizes GL_ε in $H^1(\Omega, \mathbb{C}) \times [A_{\mathrm{ex}} + H_{\mathrm{curl}}]$ and satisfies the following properties as $\varepsilon \to 0$:

(a) $GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, h_{\mathrm{ex}} A_\varepsilon^0) + o(1) \leq GL_\varepsilon(\mathsf{u}, \mathsf{A}) \leq GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, h_{\mathrm{ex}} A_\varepsilon^0)$.

(b) $\|1 - |u|\|_{L^\infty(\Omega)} = o(1)$ *and* $\||\mathsf{u}| - \rho_\varepsilon\|_{L^\infty(\Omega)} = o(1)$.

(c) *There exists $\delta > 0$ such that*

$$F_{\varepsilon,\Omega}(u, A) \leq C\varepsilon^{1+\delta}.$$

(d) $\|\mu(u, A)\|_{C_T^{0,1}(\Omega,\mathbb{R}^3)^*} = o(1)$ *and* $\|\mu(\rho_\varepsilon^{-1}\mathsf{u}, \mathsf{A})\|_{C_T^{0,1}(\Omega,\mathbb{R}^3)^*} = o(1)$.

(e) *Up to a gauge transformation, we have*

$$\inf_{\theta \in [0,2\pi]} \|u - e^{i\theta}\|_{H^1(\Omega,\mathbb{C})} + \|A\|_{\dot{H}^1_{\mathrm{div}=0}} = o(1).$$

(f) *Up to a gauge transformation, $(\rho_\varepsilon^{-1}\mathsf{u}, \mathsf{A})$ converges to $(e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, h_{\mathrm{ex}} A_\varepsilon^0)$. More precisely, we have*

$$\inf_{\theta \in [0,2\pi]} \|\rho_\varepsilon^{-1}\mathsf{u} - e^{i\theta} e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}\|_{H^1(\Omega,\mathbb{C})} + \|\mathsf{A} - h_{\mathrm{ex}} A_\varepsilon^0\|_{\dot{H}^1_{\mathrm{div}=0}} = o(1).$$

Remark 1.1. *The condition $p_0 > \frac{3}{1-3\alpha}$ approaches the natural threshold $p_0 > 3$ as $\alpha \to 0$. The requirement $p_0 > 3$ is related to the Hölder regularity needed to estimate the vorticity term. More precisely, the argument uses*

$$W^{1,p_0}(\Omega) \hookrightarrow C^{0,\beta_0}(\overline{\Omega}), \qquad \beta_0 = 1 - \frac{3}{p_0},$$

together with a quantitative vorticity estimate that provides a gain of order $\varepsilon^{\frac{\beta_0}{3}}$. The condition $\frac{\beta_0}{3} > \alpha$ is exactly equivalent to the condition on p_0 above.

Our last result concerns the uniqueness, up to a gauge transformation, of locally minimizing configurations satisfying a suitable free-energy bound after factoring out the Meissner state. The vortexless character of the configuration is not assumed, but instead follows from the hypotheses. In particular, this result applies to the locally minimizing branch constructed in Theorem 1.1.

Theorem 1.2. *Let $\alpha \in (0, 1)$ and suppose that*

$$p_0 > \frac{3}{1-\alpha}.$$

There exist constants $c_0 > 0$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$, if $h_{\mathrm{ex}} \leq \varepsilon^{-\alpha}$, then a configuration $(\mathsf{u}, \mathsf{A}) = (\rho_\varepsilon u e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, A + h_{\mathrm{ex}} A_\varepsilon^0)$ which locally minimizes GL_ε in $H^1(\Omega, \mathbb{C}) \times [A_{\mathrm{ex}} + H_{\mathrm{curl}}]$ and satisfies $F_{\varepsilon,\Omega}(u, A) \leq c_0\varepsilon$ is unique up to a gauge transformation.

Remark 1.2. *Theorem 1.2 applies, in particular, to the locally minimizing configuration constructed in Theorem 1.1. Indeed, for some $\delta > 0$,*

$$F_{\varepsilon,\Omega}(u, A) \leq C\varepsilon^{1+\delta} = o(\varepsilon).$$

Consequently, for any fixed $c_0 > 0$, we have

$$F_{\varepsilon,\Omega}(u, A) \leq c_0\varepsilon$$

for all sufficiently small ε. Moreover, in the range $\alpha \in (0, \frac{1}{4})$ covered by Theorem 1.1,

$$\frac{3}{1-3\alpha} > \frac{3}{1-\alpha},$$

so the condition on p_0 required for existence automatically implies the one required for uniqueness. Therefore, the locally minimizing branch constructed in Theorem 1.1 is unique up to a gauge transformation without any additional assumption on p_0.

For this common range of existence and uniqueness, the required exponent approaches the threshold $p_0 > 3$ as $\alpha \to 0$, which is natural in view of the Hölder embedding

$$W^{1,p_0}(\Omega) \hookrightarrow C^{0,1-\frac{3}{p_0}}(\overline{\Omega}).$$

On the other hand, the required exponent approaches 12 as $\alpha \to \frac{1}{4}$.

Together, Theorems 1.1 and 1.2 provide a rigorous description of the persistence of the Meissner branch in the three-dimensional pinned setting. They show that locally minimizing vortexless configurations continue to exist, and remain unique within a natural quantitatively controlled class, for applied fields extending far beyond the first critical field. In particular, the loss of global minimality at H_{c_1} does not correspond to the disappearance of the Meissner branch. This is consistent with the expected superheating scenario, according to which this branch should remain locally stable until a much larger applied field, of order ε^{-1}.

These results constitute, to the best of our knowledge, the first rigorous analysis of the persistence and uniqueness of the Meissner branch for the three-dimensional magnetic Ginzburg–Landau functional with a spatially dependent pinning term. Although the ranges obtained here do not reach the expected scale of the superheating field, they provide an algebraic regime of locally stable vortexless solutions and extend the variational picture previously established for the homogeneous model to the pinned setting.

Acknowledgments. This work was partially funded by ANID FONDECYT 1231593.

AI declaration. The mathematical content of this paper, including its results and proof strategies, was developed by the author. Microsoft 365 Copilot, based on the GPT-5 reasoning model, was used to assist with the revision of the manuscript, including language editing, improvements in clarity and organization, and checks of notation, consistency, and selected mathematical calculations. The author carefully reviewed and independently verified all AI-assisted changes and takes full responsibility for the content of the manuscript.

2. Preliminaries

In this section, we collect some preliminary results that will be used throughout the paper. We first introduce the different gauge choices needed in our analysis and recall some classical estimates. We then establish some basic properties of the pinning profile ρ_ε and of critical points of the Ginzburg–Landau energy. Finally, we recall a useful splitting of the energy and prove the clearing-out and vorticity estimates that will play an important role in the proofs of our main results.

2.1. Gauge choices.

We begin by introducing the three gauge choices that will be used throughout the paper. The first two are defined in bounded domains, while the third one is a global divergence-free gauge in $\mathbb{R}^3$.

Definition 2.1. *We say that a vector field A is in the Coulomb gauge in a bounded domain Θ if*

$$\left\{ \begin{array}{rcll} \operatorname{div} A & = & 0 & \text{in } \Theta \\ A \cdot \nu & = & 0 & \text{on } \partial\Theta. \end{array} \right. \tag{2.1}$$

Every vector field $A \in L^2(\Theta, \mathbb{R}^3)$ can be put in this gauge. Indeed, there exists a function $\phi \in H^1(\Theta)$, unique up to an additive constant, characterized by

$$\int_\Theta \nabla\phi \cdot \nabla\psi = \int_\Theta A \cdot \nabla\psi, \quad \forall \psi \in H^1(\Theta).$$

Equivalently, ϕ is the weak solution of

$$\left\{ \begin{array}{rcll} \Delta\phi & = & \operatorname{div} A & \text{in } \Theta \\ \dfrac{\partial\phi}{\partial\nu} & = & A \cdot \nu & \text{on } \partial\Theta. \end{array} \right.$$

It follows that $A - \nabla\phi$ satisfies (2.1). After extending ϕ to a function in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$, the corresponding gauge transformation is defined in the whole space.

The presence of the pinning profile ρ_ε leads naturally to the following weighted version of the Coulomb gauge, which arises in the construction of the Meissner state and in the estimates for the phase.

Definition 2.2. *We say that A is in the weighted Coulomb gauge associated with the weight ρ_ε^2 if*

$$\left\{ \begin{array}{rcll} \operatorname{div}(\rho_\varepsilon^2 A) & = & 0 & \text{in } \Omega \\ A \cdot \nu & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2.2}$$

As in the previous case, every vector field $A \in L^2(\Omega, \mathbb{R}^3)$ can be put in this gauge. Indeed, since $b \leq \rho_\varepsilon^2 \leq 1$, there exists a function $\phi \in H^1(\Omega)$, unique up to an additive constant, characterized by

$$\int_\Omega \rho_\varepsilon^2 \nabla\phi \cdot \nabla\psi = \int_\Omega \rho_\varepsilon^2 A \cdot \nabla\psi, \quad \forall \psi \in H^1(\Omega).$$

Equivalently, ϕ is the weak solution of

$$\left\{ \begin{array}{rcll} \operatorname{div}(\rho_\varepsilon^2 \nabla\phi) & = & \operatorname{div}(\rho_\varepsilon^2 A) & \text{in } \Omega \\ \dfrac{\partial\phi}{\partial\nu} & = & A \cdot \nu & \text{on } \partial\Omega. \end{array} \right. \tag{2.3}$$

It follows that $A-\nabla\phi$ satisfies (2.2). As before, ϕ may be extended to a function in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$, so that the corresponding gauge transformation is defined in the whole space.

Remark 2.1. *As discussed in the introduction, the regularity of the solution ϕ of (2.3) depends on the regularity and oscillation of the weight ρ_ε^2. Under assumption (1.11), if $A \in L^{p_0}(\Omega,\mathbb{R}^3)$, then* [BW05, Theorem 2] *yields*

$$\|\nabla\phi\|_{L^{p_0}(\Omega,\mathbb{R}^3)} \le C\|A\|_{L^{p_0}(\Omega,\mathbb{R}^3)},$$

where C is independent of ε.

For the global gauge choice, we recall some additional properties of the homogeneous Sobolev space introduced in the previous section. By Sobolev embedding, there exists a constant $C>0$ such that

$$\|A\|_{L^6(\mathbb{R}^3,\mathbb{R}^3)} \le C\|\nabla A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \tag{2.4}$$

for every $A \in \dot H^1(\mathbb{R}^3,\mathbb{R}^3)$. Moreover, by [KS91, Proposition 2.4],

$$\dot H^1(\mathbb{R}^3,\mathbb{R}^3) = \left\{A \in L^6(\mathbb{R}^3,\mathbb{R}^3) \;\middle|\; \nabla A \in L^2(\mathbb{R}^3,\mathbb{R}^3)\right\}.$$

In particular, combining (2.4) and (1.2), for every $A \in \dot H^1_{\mathrm{div}=0}$ we have

$$\|A\|_{L^6(\mathbb{R}^3,\mathbb{R}^3)} \le C\|\operatorname{curl} A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}.$$

We finally introduce the global divergence-free gauge.

Definition 2.3. *We say that a vector field A is in the global divergence-free gauge if*

$$A \in \dot H^1_{\mathrm{div}=0}.$$

The fact that every configuration can be put in this gauge follows from the decomposition established in [Rom19a, Lemma 2.1].

Lemma A. *Every vector field $A \in H_{\mathrm{curl}}$ admits a decomposition*

$$A = \operatorname{curl} B + \nabla\Phi,$$

where $B, \operatorname{curl} B \in \dot H^1_{\mathrm{div}=0}$ and $\Phi \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$.

Indeed, after the gauge transformation

$$u \to ue^{-i\Phi}, \qquad A \to A - \nabla\Phi,$$

we obtain $A = \operatorname{curl} B \in \dot H^1_{\mathrm{div}=0}$.

2.2. **Some classical estimates.** We will repeatedly use the following classical estimates for vector fields in the Coulomb gauge, which follow directly from [AS13, Corollaries 3.4 and 3.5].

Lemma B. *Let Θ be a bounded and simply connected domain of $\mathbb{R}^3$ with C^2 boundary, and let A satisfy (2.1) in Θ. Then, for any $p \in (1,\infty)$,*

$$\|A\|_{W^{1,p}(\Theta,\mathbb{R}^3)} \le C\|\operatorname{curl} A\|_{L^p(\Theta,\mathbb{R}^3)}$$
$$\|A\|_{W^{2,p}(\Theta,\mathbb{R}^3)} \le C\|\operatorname{curl} A\|_{W^{1,p}(\Theta,\mathbb{R}^3)}.$$

Here, C depends only on Θ and p.

We will also use the following estimate, which is the Neumann counterpart of [BBH93, Lemma A.2]. The proof follows the same argument, with only minor modifications.

Lemma C. *Let Θ be a bounded domain of $\mathbb{R}^N$, with $N \geq 2$, having C^2 boundary. If v satisfies*

$$\left\{ \begin{array}{rcll} -\Delta v & = & f & \text{in } \Theta \\ \frac{\partial v}{\partial \nu} & = & 0 & \text{on } \partial\Theta, \end{array} \right.$$

where f is such that $\int_\Theta f = 0$, then there exists a constant $C > 0$ depending only on Θ such that

$$\|\nabla v\|^2_{L^\infty(\Theta)} \leq C\|v\|_{L^\infty(\Theta)} \left(\|\Delta v\|_{L^\infty(\Theta)} + \|v\|_{L^\infty(\Theta)} \right).$$

2.3. **Properties of the pinning profile.** We next establish some basic properties of the pinning profile ρ_ε defined by (1.3).

Proposition 2.1. *We have $b \leq \rho_\varepsilon^2 \leq 1$ and*

$$\|\nabla \rho_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C\varepsilon^{-1}.$$

Proof. Testing (1.3) against $(\rho_\varepsilon - 1)_+ \in H^1(\Omega)$, we find

$$\int_{\{\rho_\varepsilon > 1\}} |\nabla \rho_\varepsilon|^2 = \frac{1}{\varepsilon^2} \int_{\{\rho_\varepsilon > 1\}} (\rho_\varepsilon - 1)\rho_\varepsilon (a_\varepsilon - \rho_\varepsilon^2) \leq 0,$$

where the inequality follows from $a_\varepsilon - \rho_\varepsilon^2 < 0$ when $\rho_\varepsilon > 1$. From this we deduce that $|\{\rho_\varepsilon > 1\}| = 0$, and hence $\rho_\varepsilon \leq 1$.

Analogously, by testing (1.3) against $(\rho_\varepsilon - \sqrt{b})_- = -\min\left\{\rho_\varepsilon - \sqrt{b}, 0\right\}$, we deduce that $\sqrt{b} \leq \rho_\varepsilon$. Therefore,

$$b \leq \rho_\varepsilon^2 \leq 1.$$

On the other hand, Lemma C, together with the bounds $b \leq \rho_\varepsilon^2 \leq 1$ and $b \leq a_\varepsilon \leq 1$, yields

$$\begin{aligned} \|\nabla \rho_\varepsilon\|^2_{L^\infty(\Omega,\mathbb{R}^3)} &\leq C\|\rho_\varepsilon\|_{L^\infty(\Omega)} \left(\|\Delta \rho_\varepsilon\|_{L^\infty(\Omega)} + \|\rho_\varepsilon\|_{L^\infty(\Omega)} \right) \\ &\leq C \left\| \frac{\rho_\varepsilon}{\varepsilon^2}(a_\varepsilon - \rho_\varepsilon^2) \right\|_{L^\infty(\Omega)} + C \\ &\leq \frac{C}{\varepsilon^2}. \end{aligned}$$

The result follows. □

2.4. **Critical points of the Ginzburg–Landau energy.** We now collect some properties of critical points of GL_ε that will be used throughout the paper.

Proposition 2.2 (Ginzburg–Landau equations)**.** *If $(\mathsf{u}, \mathsf{A}) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ is a critical point of GL_ε, then (u, A) satisfies the system of equations*

$$\left\{ \begin{array}{rcll} -(\nabla_{\mathsf{A}})^2 \mathsf{u} & = & \frac{1}{\varepsilon^2}\mathsf{u}(a_\varepsilon - |\mathsf{u}|^2) & \text{in } \Omega \\ \operatorname{curl}(\mathsf{H} - H_{\text{ex}}) & = & (i\mathsf{u}, \nabla_{\mathsf{A}}\mathsf{u})\chi_\Omega & \text{in } \mathbb{R}^3 \\ \nabla_{\mathsf{A}}\mathsf{u} \cdot \nu & = & 0 & \text{on } \partial\Omega \\ [\mathsf{H} - H_{\text{ex}}] \times \nu & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2.5}$$

Here, χ_Ω is the characteristic function of Ω, $[\,\cdot\,]$ denotes the jump across $\partial\Omega$,

$$\nabla_{\mathsf{A}}\mathsf{u}\cdot\nu=\sum_{j=1}^{3}(\partial_j\mathsf{u}-i\mathsf{A}_j\mathsf{u})\nu_j,$$

and the covariant Laplacian $(\nabla_{\mathsf{A}})^2$ is defined by

$$(\nabla_{\mathsf{A}})^2\mathsf{u}=(\operatorname{div}-i\mathsf{A}\cdot)\nabla_{\mathsf{A}}\mathsf{u}.$$

Proof. The proof follows by standard variations of GL_ε with respect to u and A and is analogous to the proof in the homogeneous case given in [Rom19a, Proposition 2.1]. □

Proposition 2.3. *Let (u,A) be a critical point of GL_ε. Then*

$$\|\mathsf{u}\|_{L^\infty(\Omega,\mathbb{C})}\leq 1\quad\text{and}\quad\left\|\frac{\mathsf{u}}{\rho_\varepsilon}\right\|_{L^\infty(\Omega,\mathbb{C})}\leq 1. \tag{2.6}$$

Moreover, if A is in the Coulomb gauge in Ω, then

$$\|\nabla\mathsf{u}\|_{L^\infty(\Omega,\mathbb{C}^3)}\leq C\left(\frac{1}{\varepsilon}+\|\mathsf{A}\|_{L^\infty(\Omega,\mathbb{R}^3)}\right). \tag{2.7}$$

Proof. By taking the inner product of the first equation in (2.5) with u and using the first boundary condition, a direct computation shows that $|\mathsf{u}|$ satisfies

$$\begin{cases}-\frac{1}{2}\Delta|\mathsf{u}|^2 &=& \frac{1}{\varepsilon^2}|\mathsf{u}|^2(a_\varepsilon-|\mathsf{u}|^2)-|\nabla_{\mathsf{A}}\mathsf{u}|^2 & \text{in }\Omega\\ \nabla|\mathsf{u}|^2\cdot\nu &=& 0 & \text{on }\partial\Omega.\end{cases}$$

Testing against $(|\mathsf{u}|^2-1)_+\in H^1(\Omega)$, we find

$$\frac{1}{2}\int_{\{|\mathsf{u}|^2>1\}}|\nabla|\mathsf{u}|^2|^2=\frac{1}{\varepsilon^2}\int_{\{|\mathsf{u}|^2>1\}}|\mathsf{u}|^2(a_\varepsilon-|\mathsf{u}|^2)(|\mathsf{u}|^2-1)-|\nabla_{\mathsf{A}}\mathsf{u}|^2(|\mathsf{u}|^2-1)\leq 0,$$

where the inequality follows from $a_\varepsilon-|\mathsf{u}|^2<0$ when $|\mathsf{u}|^2>1$. From this we deduce that $|\{|\mathsf{u}|^2>1\}|=0$, and hence $|\mathsf{u}|^2\leq 1$ in Ω.

Now, writing $u=\rho_\varepsilon^{-1}\mathsf{u}$, by combining the first equation and boundary condition in (2.5) with (1.3), we find

$$\begin{cases}-(\operatorname{div}-i\mathsf{A}\cdot)(\rho_\varepsilon^2\nabla_{\mathsf{A}}u) &=& \frac{1}{\varepsilon^2}\rho_\varepsilon^4u(1-|u|^2) & \text{in }\Omega\\ \nabla_{\mathsf{A}}u\cdot\nu &=& 0 & \text{on }\partial\Omega.\end{cases}$$

Taking the inner product with u, a direct computation shows that $|u|$ satisfies

$$\begin{cases}-\frac{1}{2}\operatorname{div}(\rho_\varepsilon^2\nabla|u|^2) &=& \frac{\rho_\varepsilon^4}{\varepsilon^2}|u|^2(1-|u|^2)-\rho_\varepsilon^2|\nabla_{\mathsf{A}}u|^2 & \text{in }\Omega\\ \nabla|u|^2\cdot\nu &=& 0 & \text{on }\partial\Omega.\end{cases}$$

Testing against $(|u|^2-1)_+\in H^1(\Omega)$, we find

$$\frac{1}{2}\int_{\{|u|^2>1\}}\rho_\varepsilon^2|\nabla|u|^2|^2=\frac{1}{\varepsilon^2}\int_{\{|u|^2>1\}}\rho_\varepsilon^4|u|^2(1-|u|^2)(|u|^2-1)-\rho_\varepsilon^2|\nabla_{\mathsf{A}}u|^2(|u|^2-1)\leq 0,$$

where the inequality follows from $1-|u|^2<0$ when $|u|^2>1$. From this we deduce that $|\{|u|^2>1\}|=0$, and hence $|u|^2\leq 1$ in Ω.

On the other hand, if A is in the Coulomb gauge in Ω, by expanding the covariant Laplacian, we find

$$-\Delta \mathsf{u} = \frac{1}{\varepsilon^2}\mathsf{u}(a_\varepsilon - |\mathsf{u}|^2) - 2i\mathsf{A}\cdot\nabla\mathsf{u} - |\mathsf{A}|^2\mathsf{u},$$

together with the boundary condition $\nabla\mathsf{u}\cdot\nu = 0$ on $\partial\Omega$. Hence, by applying Lemma C and using $\|\mathsf{u}\|_{L^\infty(\Omega,\mathbb{C})} \leq 1$, we find

$$\|\nabla\mathsf{u}\|^2_{L^\infty(\Omega,\mathbb{C}^3)} \leq C\left(\frac{1}{\varepsilon^2} + \|\mathsf{A}\|_{L^\infty(\Omega,\mathbb{R}^3)}\|\nabla\mathsf{u}\|_{L^\infty(\Omega,\mathbb{C}^3)} + \|\mathsf{A}\|^2_{L^\infty(\Omega,\mathbb{R}^3)}\right),$$

from where it follows that

$$\|\nabla\mathsf{u}\|_{L^\infty(\Omega,\mathbb{C}^3)} \leq C\left(\frac{1}{\varepsilon} + \|\mathsf{A}\|_{L^\infty(\Omega,\mathbb{R}^3)}\right).$$

□

Proposition 2.4. *Let (u,A) be a critical point of GL_ε. If A is in the Coulomb gauge in Ω, then*

$$\|\mathsf{A}\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C\left(\left\|\nabla_{\mathsf{A}}\left(\frac{\mathsf{u}}{\rho_\varepsilon}\right)\right\|_{L^2(\Omega,\mathbb{C}^3)} + h_{\mathrm{ex}}\right).$$

Proof. Since (u,A) is a critical point of GL_ε, it satisfies the second Ginzburg–Landau equation

$$\operatorname{curl}^2(\mathsf{A} - A_{\mathrm{ex}}) = (i\mathsf{u}, \nabla_{\mathsf{A}}\mathsf{u})\chi_\Omega \quad \text{in } \mathbb{R}^3. \tag{2.8}$$

Since $\mathsf{A} - A_{\mathrm{ex}} \in H_{\mathrm{curl}}$, we have

$$\operatorname{curl}(\mathsf{A} - A_{\mathrm{ex}}) \in L^2(\mathbb{R}^3,\mathbb{R}^3).$$

Moreover,

$$\operatorname{div}\operatorname{curl}(\mathsf{A} - A_{\mathrm{ex}}) = 0 \quad \text{in } \mathbb{R}^3,$$

and, by (2.8), (2.6), and the fact that $\nabla_{\mathsf{A}}\mathsf{u} \in L^2(\Omega,\mathbb{C}^3)$,

$$\operatorname{curl}^2(\mathsf{A} - A_{\mathrm{ex}}) \in L^2(\mathbb{R}^3,\mathbb{R}^3).$$

Using the classical whole-space div–curl identity

$$\|\nabla\operatorname{curl}(\mathsf{A} - A_{\mathrm{ex}})\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^3)} = \|\operatorname{curl}^2(\mathsf{A} - A_{\mathrm{ex}})\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^3)} + \|\operatorname{div}\operatorname{curl}(\mathsf{A} - A_{\mathrm{ex}})\|^2_{L^2(\mathbb{R}^3)},$$

we obtain

$$\operatorname{curl}(\mathsf{A} - A_{\mathrm{ex}}) \in H^1(\mathbb{R}^3,\mathbb{R}^3)$$

and

$$\|\nabla\operatorname{curl}(\mathsf{A} - A_{\mathrm{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} = \|\operatorname{curl}^2(\mathsf{A} - A_{\mathrm{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}.$$

Since

$$\operatorname{curl}(\mathsf{A} - A_{\mathrm{ex}}) \in H^1(\mathbb{R}^3,\mathbb{R}^3) \subset \dot{H}^1(\mathbb{R}^3,\mathbb{R}^3)$$

and this vector field is divergence free, we also have

$$\operatorname{curl}(\mathsf{A} - A_{\mathrm{ex}}) \in \dot{H}^1_{\mathrm{div}=0}.$$

Writing $u = \rho_\varepsilon^{-1}\mathsf{u}$, we observe that

$$\begin{aligned}(i\mathsf{u}, \nabla_{\mathsf{A}}\mathsf{u}) = (i\rho_\varepsilon u, \nabla_{\mathsf{A}}(\rho_\varepsilon u)) &= \rho_\varepsilon(iu, u\nabla\rho_\varepsilon + \rho_\varepsilon\nabla_{\mathsf{A}}u)\\ &= \rho_\varepsilon\nabla\rho_\varepsilon(iu,u) + \rho_\varepsilon^2(iu,\nabla_{\mathsf{A}}u) = \rho_\varepsilon^2(iu,\nabla_{\mathsf{A}}u).\end{aligned}$$

Using (2.8), (2.6), and the fact that $\rho_\varepsilon \leq 1$, we obtain

$$\|\nabla \operatorname{curl}(\mathsf{A} - A_{\rm ex})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} = \|\operatorname{curl}^2(\mathsf{A} - A_{\rm ex})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \leq \|\nabla_{\mathsf{A}} u\|_{L^2(\Omega,\mathbb{C}^3)}.$$

In view of (2.4), we then find that

$$\|\operatorname{curl}(\mathsf{A} - A_{\rm ex})\|_{L^6(\mathbb{R}^3,\mathbb{R}^3)} \leq C\|\nabla_{\mathsf{A}} u\|_{L^2(\Omega,\mathbb{C}^3)}.$$

Using Lemma B, and recalling that both A and $A_{\rm ex}$ are in the Coulomb gauge in Ω, we conclude that

$$\|\mathsf{A} - A_{\rm ex}\|_{W^{1,6}(\Omega,\mathbb{R}^3)} \leq C\|\operatorname{curl}(\mathsf{A} - A_{\rm ex})\|_{L^6(\Omega,\mathbb{R}^3)}$$

and

$$\|A_{\rm ex}\|_{W^{1,6}(\Omega,\mathbb{R}^3)} \leq C\|\operatorname{curl} A_{\rm ex}\|_{L^6(\Omega,\mathbb{R}^3)}.$$

We then deduce that

$$\|\mathsf{A}\|_{W^{1,6}(\Omega,\mathbb{R}^3)} \leq C\left(\|\nabla_{\mathsf{A}} u\|_{L^2(\Omega,\mathbb{C}^3)} + \|\operatorname{curl} A_{\rm ex}\|_{L^6(\Omega,\mathbb{R}^3)}\right).$$

Finally, by Sobolev embedding and recalling that $\operatorname{curl} A_{\rm ex} = h_{\rm ex} H_{0,\rm ex}$, we find

$$\|\mathsf{A}\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C\left(\|\nabla_{\mathsf{A}} u\|_{L^2(\Omega,\mathbb{C}^3)} + h_{\rm ex}\right).$$

This concludes the proof. □

2.5. **Energy and vorticity estimates.** We conclude this section with three ingredients that will play a central role in the proofs of our main results. We first recall the energy splitting established in [DVR26a, Proposition 1.1], and then prove a clearing-out estimate and a quantitative estimate for the vorticity.

Proposition A (Energy splitting)**.** *Given any configuration* $(\mathsf{u},\mathsf{A}) \in H^1(\Omega,\mathbb{C})\times[A_{\rm ex}+H_{\rm curl}]$*, letting* (u,A) *be defined via the relation*

$$(\mathsf{u},\mathsf{A}) = (\rho_\varepsilon u e^{ih_{\rm ex}\phi^0_\varepsilon}, A + h_{\rm ex}A^0_\varepsilon),$$

we have

$$GL_\varepsilon(\mathsf{u},\mathsf{A}) = GL_\varepsilon(\rho_\varepsilon e^{ih_{\rm ex}\phi^0_\varepsilon}, h_{\rm ex}A^0_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\rm ex}\int_\Omega \mu(u,A)\cdot B^0_\varepsilon + R_0, \tag{2.9}$$

where

$$F_{\varepsilon,\rho_\varepsilon}(u,A) = \frac{1}{2}\int_\Omega \left(\rho_\varepsilon^2|\nabla_A u|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2\right) + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl} A|^2$$

and

$$R_0 := \frac{h_{\rm ex}^2}{2}\int_\Omega \frac{|\operatorname{curl} B^0_\varepsilon|^2}{\rho_\varepsilon^2}(|u|^2-1).$$

Lemma 2.1 (Clearing-out estimate)**.** *Let* $u \in H^1(\Omega,\mathbb{C})$ *satisfy* $\|\nabla|u|\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C\varepsilon^{-1}$*. Then*

$$(1-|u|(x_0))^5 \leq \frac{C}{\varepsilon^3}\int_{B_\varepsilon(x_0)\cap\Omega}(1-|u|^2)^2, \quad \forall x_0 \in \overline{\Omega}.$$

Proof. The proof is an adaptation of [BBO01, Lemma III.3]. Let

$$k := 1 - |u|(x_0).$$

We may assume that $k > 0$, since otherwise the result trivially holds. For any $x \in \Omega$, we have

$$\big||u|(x) - |u|(x_0)\big| \leq \|\nabla |u|\|_{L^\infty(\Omega,\mathbb{R}^3)}|x - x_0| \leq \frac{C}{\varepsilon}|x - x_0|.$$

Letting $r := \frac{k\varepsilon}{2C}$, we obtain

$$1 - |u|(x) \geq k - \frac{C}{\varepsilon}|x - x_0| \geq \frac{k}{2} \quad \forall x \in B_r(x_0) \cap \Omega.$$

Since $0 < k \leq 1$, by increasing C if necessary, we have $r < \varepsilon$, and hence $B_r(x_0) \subset B_\varepsilon(x_0)$.

The C^2 regularity of $\partial\Omega$ implies that there exists a constant $c > 0$, depending only on Ω, such that

$$|B_r(x_0) \cap \Omega| \geq cr^3.$$

Hence

$$\begin{aligned}\int_{B_\varepsilon(x_0)\cap\Omega}(1 - |u|^2)^2 \geq \int_{B_r(x_0)\cap\Omega}(1 - |u|^2)^2 &= \int_{B_r(x_0)\cap\Omega}(1 - |u|)^2(1 + |u|)^2 \\ &\geq \int_{B_r(x_0)\cap\Omega}(1 - |u|)^2 \geq c\frac{k^2}{4}r^3 = \frac{c}{32C^3}k^5\varepsilon^3.\end{aligned}$$

The inequality is thus proved. □

Proposition 2.5 (Quantitative vorticity estimate)**.** *Let $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ be such that $F_{\varepsilon,\Omega}(u, A) \leq \varepsilon^\lambda$, for some $\lambda \in (0, 1]$, and let $\beta \in (0, 1]$.*

If $\lambda > \frac{1}{3}$, there exists $C_\beta > 0$ depending on β and Ω such that

$$\|\mu(u, A)\|_{C_T^{0,\beta}(\Omega,\mathbb{R}^3)^*} \leq C_\beta \varepsilon^{\frac{\beta}{3}} F_{\varepsilon,\Omega}(u, A).$$

If $\lambda \leq \frac{1}{3}$, then, for any $\gamma \in (0, \lambda)$, there exists $C_\beta > 0$ depending on β and Ω such that

$$\|\mu(u, A)\|_{C_T^{0,\beta}(\Omega,\mathbb{R}^3)^*} \leq C_\beta \varepsilon^{\beta(\lambda-\gamma)} F_{\varepsilon,\Omega}(u, A).$$

The estimates hold for all sufficiently small ε, where the corresponding threshold may depend on λ and, in the second case, on γ.

Proof. We consider, for ε sufficiently small and $\delta = \delta(\varepsilon) = \varepsilon^\sigma$, with $\sigma \leq \frac{1+\lambda}{4}$ to be fixed later, the grid $\mathfrak{G}(b_\varepsilon, \delta)$ associated with (u, A) by [Rom19b, Lemma 2.1]. In particular, using the same notation as in that lemma, we have

$$|u| > 5/8 \quad \text{on } \mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, \delta)),$$

$$I_\varepsilon^1 := \int_{\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon,\delta))} e_\varepsilon(u, A)d\mathcal{H}^1 \leq C\delta^{-2}F_{\varepsilon,\Omega}(u, A) \leq C\varepsilon^{-2\sigma+\lambda},$$

$$I_\varepsilon^2 := \int_{\mathfrak{R}_2(\mathfrak{G}(b_\varepsilon,\delta))} e_\varepsilon(u, A)d\mathcal{H}^2 \leq C\delta^{-1}F_{\varepsilon,\Omega}(u, A) \leq C\varepsilon^{-\sigma+\lambda}. \tag{2.10}$$

Here, C is a universal constant.

We claim that if $\sigma < \lambda$ and ε is sufficiently small, then, for each face ω of a cube of the grid, every connected component S_ω of

$$\{x \in \omega \mid |u(x)| \le 1/2\}$$

satisfies

$$d_{S_\omega} := \deg(u/|u|, \partial S_\omega) = 0.$$

Indeed, by [Rom19b, Lemma 4.1], a result adapted from [Jer99], we have

$$|d_{S_\omega}| \le C \int_{S_\omega} |\nabla_A u|^2,$$

where C is a universal constant. Combining this with (2.10), we obtain

$$|d_{S_\omega}| \le C I_\varepsilon^2 \le C\varepsilon^{-\sigma+\lambda}.$$

Since d_{S_ω} is an integer and $\sigma < \lambda$, the claim follows for all sufficiently small ε.

It follows that the vorticity approximation ν_ε of $\mu(u, A)$ provided by [Rom19b, Theorem 1.1] vanishes identically in $\overline{\Omega}$. Estimate [Rom19b, (8.10)] then yields

$$\|\mu(u,A)\|_{C_T^{0,1}(\Omega,\mathbb{R}^3)^*} \le C\delta F_{\varepsilon,\Omega}(u,A) + C\max\{\varepsilon\delta^{-1}F_{\varepsilon,\Omega}(u,A), \varepsilon\}(1+\delta^{-2})F_{\varepsilon,\Omega}(u,A).$$

Notice that

$$\varepsilon\delta^{-1}F_{\varepsilon,\Omega}(u,A) \le \varepsilon\varepsilon^{-\sigma}\varepsilon^{\lambda} < \varepsilon$$

since $\sigma < \lambda$. Therefore, the previous estimate gives

$$\|\mu(u,A)\|_{C_T^{0,1}(\Omega,\mathbb{R}^3)^*} \le C(\delta + \varepsilon\delta^{-2})F_{\varepsilon,\Omega}(u,A)$$

and hence

$$\|\mu(u,A)\|_{C_T^{0,1}(\Omega,\mathbb{R}^3)^*} \le C\varepsilon^{\min\{\sigma,1-2\sigma\}}F_{\varepsilon,\Omega}(u,A). \tag{2.11}$$

We first consider the case $\lambda > \frac{1}{3}$. Choosing $\sigma = \frac{1}{3}$, we have $\sigma < \lambda$ and

$$\sigma \le \frac{1+\lambda}{4}.$$

Moreover,

$$\min\{\sigma, 1-2\sigma\} = \frac{1}{3}.$$

Thus, (2.11) yields

$$\|\mu(u,A)\|_{C_T^{0,1}(\Omega,\mathbb{R}^3)^*} \le C\varepsilon^{\frac{1}{3}}F_{\varepsilon,\Omega}(u,A). \tag{2.12}$$

Suppose now that $\lambda \le \frac{1}{3}$. Given $\gamma \in (0, \lambda)$, we choose

$$\sigma = \lambda - \gamma.$$

Then $\sigma < \lambda$, $\sigma < \frac{1}{3}$, and

$$\sigma \le \frac{1+\lambda}{4}.$$

Hence,

$$\min\{\sigma, 1-2\sigma\} = \sigma = \lambda - \gamma,$$

and (2.11) yields

$$\|\mu(u,A)\|_{C_T^{0,1}(\Omega,\mathbb{R}^3)^*} \le C\varepsilon^{\lambda-\gamma}F_{\varepsilon,\Omega}(u,A). \tag{2.13}$$

In addition, from [Rom19b, Lemma 8.1] we have

$$\|\mu(u,A)\|_{C^0(\Omega,\mathbb{R}^3)^*} \leq CF_{\varepsilon,\Omega}(u,A). \tag{2.14}$$

Interpolating between (2.12) and (2.14), we conclude that, if $\lambda > \frac{1}{3}$, then

$$\|\mu(u,A)\|_{C_T^{0,\beta}(\Omega,\mathbb{R}^3)^*} \leq C_\beta \varepsilon^{\frac{\beta}{3}} F_{\varepsilon,\Omega}(u,A).$$

On the other hand, interpolating between (2.13) and (2.14), we conclude that, if $\lambda \leq \frac{1}{3}$, then, for any $\gamma \in (0,\lambda)$,

$$\|\mu(u,A)\|_{C_T^{0,\beta}(\Omega,\mathbb{R}^3)^*} \leq C_\beta \varepsilon^{\beta(\lambda-\gamma)} F_{\varepsilon,\Omega}(u,A).$$

The proposition is thus proved. □

3. Existence

We now present the proof of Theorem 1.1.

Proof. We proceed in several steps.

Step 1. Existence of a constrained minimizer.

Given a configuration (u,A), we let (u,A) be defined by the relation

$$(\mathsf{u},\mathsf{A}) = (\rho_\varepsilon u e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, A + h_{\mathrm{ex}}A_\varepsilon^0).$$

We then consider the set

$$U = \left\{(\mathsf{u},\mathsf{A}) \in H^1(\Omega,\mathbb{C}) \times [A_{\mathrm{ex}} + H_{\mathrm{curl}}] \;\middle|\; F_{\varepsilon,\Omega}(u,A) < \varepsilon\right\}.$$

We note that $(\mathsf{u}_\circ,\mathsf{A}_\circ) := (\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, h_{\mathrm{ex}}A_\varepsilon^0)$, that is, $(u_\circ,A_\circ) = (1,0)$, satisfies $F_{\varepsilon,\Omega}(u_\circ,A_\circ) = 0$. Therefore, U is non-empty.

Let us prove that GL_ε possesses a minimizer in $\overline{U}$. For this, let $\{(\tilde{\mathsf{u}}_n,\tilde{\mathsf{A}}_n)\}_n \subset U$ be a minimizing sequence. Lemma A yields a gauge-transformed sequence

$$\{(\mathsf{u}_n,\mathsf{A}_n)\}_n \subset H^1(\Omega,\mathbb{C}) \times [A_{\mathrm{ex}} + \dot{H}^1_{\mathrm{div}=0}].$$

By gauge invariance,

$$GL_\varepsilon(\tilde{\mathsf{u}}_n,\tilde{\mathsf{A}}_n) = GL_\varepsilon(\mathsf{u}_n,\mathsf{A}_n).$$

Since $(\mathsf{u}_\circ,\mathsf{A}_\circ) \in U$, we may choose the minimizing sequence so that

$$GL_\varepsilon(\mathsf{u}_n,\mathsf{A}_n) \leq GL_\varepsilon(\mathsf{u}_\circ,\mathsf{A}_\circ)$$

for every n. Then

$$\|\nabla_{\mathsf{A}_n}\mathsf{u}_n\|_{L^2(\Omega,\mathbb{C}^3)}, \quad \|a_\varepsilon - |\mathsf{u}_n|^2\|_{L^2(\Omega)}, \quad \text{and} \quad \|\operatorname{curl}(\mathsf{A}_n - A_{\mathrm{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$$

are bounded independently of n. Since $\mathsf{A}_n - A_{\mathrm{ex}} \in \dot{H}^1_{\mathrm{div}=0}$, we have

$$\|\mathsf{A}_n - A_{\mathrm{ex}}\|_{\dot{H}^1_{\mathrm{div}=0}} = \|\operatorname{curl}(\mathsf{A}_n - A_{\mathrm{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)},$$

and therefore $\{\mathsf{A}_n - A_{\mathrm{ex}}\}_n$ is bounded in $\dot{H}^1_{\mathrm{div}=0}$. Moreover, by Sobolev embedding, $\{\mathsf{A}_n\}_n$ is bounded in $L^6(\Omega,\mathbb{R}^3)$.

On the other hand, using

$$|\mathsf{u}_n|^4 = (|\mathsf{u}_n|^2 - a_\varepsilon + a_\varepsilon)^2 \leq 2(a_\varepsilon - |\mathsf{u}_n|^2)^2 + 2a_\varepsilon^2 \leq 2(a_\varepsilon - |\mathsf{u}_n|^2)^2 + 2,$$

we get

$$\|\mathsf{u}_n\|^4_{L^4(\Omega,\mathbb{C})} \leq 2|\Omega| + 8\varepsilon^2 \int_\Omega \frac{(a_\varepsilon - |\mathsf{u}_n|^2)^2}{4\varepsilon^2},$$

which implies that $\{\mathsf{u}_n\}_n$ is bounded in $L^4(\Omega,\mathbb{C})$.

Noting that

$$\nabla \mathsf{u}_n = \nabla_{\mathsf{A}_n} \mathsf{u}_n + i\mathsf{A}_n \mathsf{u}_n,$$

and using Hölder's inequality, we deduce that $\{\mathsf{u}_n\}_n$ is bounded in $H^1(\Omega,\mathbb{C})$.

We may then extract a subsequence, still denoted by $\{(\mathsf{u}_n, \mathsf{A}_n)\}_n$, such that

$$(\mathsf{u}_n, \mathsf{A}_n - A_{\mathrm{ex}}) \rightharpoonup (\mathsf{u}, \mathsf{A} - A_{\mathrm{ex}})$$

weakly in $H^1(\Omega,\mathbb{C}) \times \dot{H}^1_{\mathrm{div}=0}$. By compact Sobolev embedding, we also have strong convergence in

$$L^q(\Omega,\mathbb{C}) \times L^q(\Omega,\mathbb{R}^3)$$

for every $q < 6$.

Let us now show that

$$GL_\varepsilon(\mathsf{u},\mathsf{A}) \leq \liminf_n GL_\varepsilon(\mathsf{u}_n,\mathsf{A}_n). \tag{3.1}$$

By strong convergence in $L^4(\Omega,\mathbb{C})$,

$$\lim_n \|a_\varepsilon - |\mathsf{u}_n|^2\|_{L^2(\Omega)} = \|a_\varepsilon - |\mathsf{u}|^2\|_{L^2(\Omega)}.$$

Also, by the weak lower semicontinuity of the norm in $\dot{H}^1_{\mathrm{div}=0}$,

$$\begin{aligned}\liminf_n \|\operatorname{curl}(\mathsf{A}_n - A_{\mathrm{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} &= \liminf_n \|\nabla(\mathsf{A}_n - A_{\mathrm{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \\ &\geq \|\nabla(\mathsf{A} - A_{\mathrm{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \\ &= \|\operatorname{curl}(\mathsf{A} - A_{\mathrm{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}.\end{aligned}$$

Moreover, $\mathsf{u}_n \to \mathsf{u}$ strongly in $L^4(\Omega,\mathbb{C})$ and $\mathsf{A}_n \to \mathsf{A}$ strongly in $L^4(\Omega,\mathbb{R}^3)$. In particular,

$$\mathsf{A}_n\mathsf{u}_n \to \mathsf{A}\mathsf{u} \quad \text{strongly in } L^2(\Omega,\mathbb{C}^3).$$

Using also the weak convergence of $\nabla \mathsf{u}_n$ in $L^2(\Omega,\mathbb{C}^3)$, we deduce that

$$\begin{aligned}\liminf_n \|\nabla_{\mathsf{A}_n}\mathsf{u}_n\|^2_{L^2(\Omega,\mathbb{C}^3)} &= \liminf_n \left(\|\nabla \mathsf{u}_n\|^2_{L^2(\Omega,\mathbb{C}^3)} - 2\int_\Omega (\nabla \mathsf{u}_n, i\mathsf{A}_n\mathsf{u}_n) + \|\mathsf{A}_n\mathsf{u}_n\|^2_{L^2(\Omega,\mathbb{C}^3)} \right) \\ &\geq \|\nabla \mathsf{u}\|^2_{L^2(\Omega,\mathbb{C}^3)} - 2\int_\Omega (\nabla \mathsf{u}, i\mathsf{A}\mathsf{u}) + \|\mathsf{A}\mathsf{u}\|^2_{L^2(\Omega,\mathbb{C}^3)} \\ &= \|\nabla_{\mathsf{A}}\mathsf{u}\|^2_{L^2(\Omega,\mathbb{C}^3)}.\end{aligned}$$

Hence, (3.1) holds.

We now claim that

$$F_{\varepsilon,\Omega}(u,A) \leq \liminf_n F_{\varepsilon,\Omega}(u_n,A_n). \tag{3.2}$$

Recall that

$$u_n = \rho_\varepsilon^{-1} e^{-ih_{\mathrm{ex}}\phi^0_\varepsilon}\mathsf{u}_n, \qquad A_n = \mathsf{A}_n - h_{\mathrm{ex}}A^0_\varepsilon \quad \text{in } \Omega.$$

We first show that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\Omega,\mathbb{C}).$$

Since $b \leq \rho_\varepsilon^2 \leq 1$ and $\rho_\varepsilon \in W^{1,\infty}(\Omega)$, multiplication by ρ_ε^{-1} is continuous in $H^1(\Omega)$. Moreover, since $p_0 > 3$, letting

$$q = \frac{2p_0}{p_0 - 2} < 6,$$

we have

$$\frac{1}{2} = \frac{1}{p_0} + \frac{1}{q}.$$

The strong convergence of u_n to u in $L^q(\Omega, \mathbb{C})$ and estimate (1.10) imply that

$$\mathsf{u}_n \nabla \phi_\varepsilon^0 \longrightarrow \mathsf{u} \nabla \phi_\varepsilon^0 \quad \text{strongly in } L^2(\Omega, \mathbb{C}^3).$$

Together with the weak convergence of u_n to u in $H^1(\Omega, \mathbb{C})$, this proves the claimed weak convergence of u_n in $H^1(\Omega, \mathbb{C})$.

On the other hand, since $A_\varepsilon^0 \in H^1(\Omega, \mathbb{R}^3)$ is independent of n, the weak convergence of A_n to A in $H^1(\Omega, \mathbb{R}^3)$ implies that

$$A_n \rightharpoonup A \quad \text{weakly in } H^1(\Omega, \mathbb{R}^3).$$

Here, we used the fact that the weak convergence of $\mathsf{A}_n - A_{\rm ex}$ to $\mathsf{A} - A_{\rm ex}$ in $\dot{H}^1_{\rm div=0}$, together with the fact that $A_{\rm ex}$ does not depend on n, implies the weak convergence of A_n to A in $H^1(\Omega, \mathbb{R}^3)$.

Arguing as in the proof of (3.1), we deduce that

$$F_{\varepsilon,\Omega}(u, A) \leq \liminf_n F_{\varepsilon,\Omega}(u_n, A_n),$$

which proves (3.2).

Moreover, by gauge invariance and since $(\tilde{\mathsf{u}}_n, \tilde{\mathsf{A}}_n) \in U$, we have

$$F_{\varepsilon,\Omega}(u_n, A_n) < \varepsilon.$$

Therefore, (3.2) implies that

$$F_{\varepsilon,\Omega}(u, A) \leq \varepsilon,$$

and hence $(\mathsf{u}, \mathsf{A}) \in \overline{U}$.

Step 2. Interior minimality and first energy estimate.

We next show that $(\mathsf{u}, \mathsf{A}) \in U$. Since (u, A) minimizes GL_ε over $\overline{U}$, this will show that it is a local minimizer of GL_ε and hence a critical point. Combining

$$GL_\varepsilon(\mathsf{u}, \mathsf{A}) \leq GL_\varepsilon(\mathsf{u}_\circ, \mathsf{A}_\circ) = GL_\varepsilon(\rho_\varepsilon e^{ih_{\rm ex}\phi_\varepsilon^0}, h_{\rm ex} A_\varepsilon^0)$$

with (2.9), we find

$$F_{\varepsilon,\rho_\varepsilon}(u, A) - h_{\rm ex} \int_\Omega \mu(u, A) \cdot B_\varepsilon^0 + R_0 \leq 0.$$

It follows that

$$F_{\varepsilon,\rho_\varepsilon}(u, A) \leq h_{\rm ex} \|\mu(u, A)\|_{C_T^{0,\beta_0}(\Omega,\mathbb{R}^3)^*} \|B_\varepsilon^0\|_{C_T^{0,\beta_0}(\Omega,\mathbb{R}^3)} - R_0, \tag{3.3}$$

where

$$\beta_0 = 1 - \frac{3}{p_0} > 0. \tag{3.4}$$

In view of (1.13), Sobolev embedding gives

$$\|B_\varepsilon^0\|_{C_T^{0,\beta_0}(\Omega,\mathbb{R}^3)} \leq C. \tag{3.5}$$

On the other hand, since $F_{\varepsilon,\Omega}(u,A)\le\varepsilon$, Proposition 2.5, applied with $\lambda=1$, yields

$$\|\mu(u,A)\|_{C_T^{0,\beta_0}(\Omega,\mathbb{R}^3)^*}\le C\varepsilon^{\frac{\beta_0}{3}}F_{\varepsilon,\Omega}(u,A). \tag{3.6}$$

We next estimate the last term in (3.3). Recalling the definition of R_0, we have

$$-R_0\le\frac{h_{\rm ex}^2}{2b}\int_\Omega|\operatorname{curl}B_\varepsilon^0|^2(1-|u|^2)_+.$$

Let

$$q=\frac{p_0}{p_0-2},$$

so that

$$\frac{2}{p_0}+\frac{1}{q}=1.$$

By Hölder's inequality and (1.12),

$$\begin{aligned}-R_0&\le Ch_{\rm ex}^2\|\operatorname{curl}B_\varepsilon^0\|_{L^{p_0}(\Omega,\mathbb{R}^3)}^2\|(1-|u|^2)_+\|_{L^q(\Omega)}\\&\le Ch_{\rm ex}^2\|(1-|u|^2)_+\|_{L^q(\Omega)}.\end{aligned}$$

If $p_0\ge4$, then $q\le2$, and hence

$$\|(1-|u|^2)_+\|_{L^q(\Omega)}\le C\|(1-|u|^2)_+\|_{L^2(\Omega)}\le C\varepsilon F_{\varepsilon,\Omega}(u,A)^{\frac12}.$$

If $3<p_0<4$, then $q>2$. Since

$$0\le(1-|u|^2)_+\le1,$$

interpolation between $L^2(\Omega)$ and $L^\infty(\Omega)$ gives

$$\begin{aligned}\|(1-|u|^2)_+\|_{L^q(\Omega)}&\le\|(1-|u|^2)_+\|_{L^2(\Omega)}^{\frac2q}\|(1-|u|^2)_+\|_{L^\infty(\Omega)}^{1-\frac2q}\\&\le C\varepsilon^{2-\frac{4}{p_0}}F_{\varepsilon,\Omega}(u,A)^{1-\frac{2}{p_0}}.\end{aligned}$$

Defining

$$\vartheta:=\min\left\{1,2-\frac{4}{p_0}\right\}, \tag{3.7}$$

we conclude in both cases that

$$-R_0\le Ch_{\rm ex}^2\varepsilon^\vartheta F_{\varepsilon,\Omega}(u,A)^{\frac\vartheta2}. \tag{3.8}$$

Combining (3.3), (3.5), (3.6), and (3.8), and recalling that $h_{\rm ex}\le\varepsilon^{-\alpha}$ and $F_{\varepsilon,\Omega}(u,A)\le\varepsilon$, we obtain

$$\begin{aligned}F_{\varepsilon,\rho_\varepsilon}(u,A)&\le Ch_{\rm ex}\varepsilon^{\frac{\beta_0}{3}}F_{\varepsilon,\Omega}(u,A)+Ch_{\rm ex}^2\varepsilon^\vartheta F_{\varepsilon,\Omega}(u,A)^{\frac\vartheta2}\\&\le C\varepsilon^{1+\frac{\beta_0}{3}-\alpha}+C\varepsilon^{\frac{3\vartheta}{2}-2\alpha}.\end{aligned}$$

Recalling (3.4), we observe that the condition

$$p_0>\frac{3}{1-3\alpha}$$

is equivalent to

$$\frac{\beta_0}{3}>\alpha.$$

Moreover, this condition implies that

$$\frac{3\vartheta}{2} - 2\alpha > 1.$$

Indeed, if $p_0 \geq 4$, then $\vartheta = 1$ and the latter inequality follows from $\alpha < \frac{1}{4}$. If $3 < p_0 < 4$, then $\vartheta = 2 - \frac{4}{p_0}$, and

$$\frac{3\vartheta}{2} - 2\alpha - 1 = 2 - \frac{6}{p_0} - 2\alpha > 0,$$

where we used $p_0 > \frac{3}{1-3\alpha} > \frac{3}{1-\alpha}$. We conclude that

$$F_{\varepsilon,\rho_\varepsilon}(u, A) = o(\varepsilon).$$

Since

$$F_{\varepsilon,\Omega}(u, A) \leq b^{-2} F_{\varepsilon,\rho_\varepsilon}(u, A),$$

we obtain

$$F_{\varepsilon,\Omega}(u, A) = o(\varepsilon).$$

Therefore, $(\mathsf{u}, \mathsf{A}) \in U$.

Recalling (2.9), an immediate consequence of the previous estimates is that

$$GL_\varepsilon(\mathsf{u}, \mathsf{A}) \geq GL_\varepsilon(\mathsf{u}_\circ, \mathsf{A}_\circ) + o(1).$$

On the other hand, since (u, A) minimizes GL_ε over $\overline{U}$ and $(\mathsf{u}_\circ, \mathsf{A}_\circ) \in U$, we have

$$GL_\varepsilon(\mathsf{u}, \mathsf{A}) \leq GL_\varepsilon(\mathsf{u}_\circ, \mathsf{A}_\circ).$$

This proves item (a).

Step 3. Clearing out and improved estimates.

Since (u, A) is a critical point of GL_ε, it satisfies the Ginzburg–Landau equations. We can gauge transform this configuration into $(\mathsf{u}^C, \mathsf{A}^C)$ in the Coulomb gauge in Ω. Notice that

$$(\rho_\varepsilon^{-1}\mathsf{u}^C, \mathsf{A}^C)$$

is gauge equivalent to

$$(u, A + h_{\mathrm{ex}}(A_\varepsilon^0 - \nabla\phi_\varepsilon^0)).$$

Hence, using (1.7), $\|\operatorname{curl} B_\varepsilon^0\|_{L^2(\Omega,\mathbb{R}^3)} \leq C$, and $\|u\|_{L^\infty(\Omega,\mathbb{C})} \leq 1$, we find

$$\begin{aligned}
\|\nabla_{\mathsf{A}^C}(\rho_\varepsilon^{-1}\mathsf{u}^C)\|^2_{L^2(\Omega,\mathbb{C}^3)} &= \|\nabla_{A+h_{\mathrm{ex}}(A_\varepsilon^0-\nabla\phi_\varepsilon^0)}u\|^2_{L^2(\Omega,\mathbb{C}^3)} \\
&= \|\nabla_{A+h_{\mathrm{ex}}\rho_\varepsilon^{-2}\operatorname{curl} B_\varepsilon^0}u\|^2_{L^2(\Omega,\mathbb{C}^3)} \\
&\leq C\left(\|\nabla_A u\|^2_{L^2(\Omega,\mathbb{C}^3)} + h_{\mathrm{ex}}^2\right) \\
&\leq C(\varepsilon + h_{\mathrm{ex}}^2).
\end{aligned}$$

Proposition 2.4 then implies that

$$\|\mathsf{A}^C\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C\varepsilon^{-\alpha}.$$

Using (2.7) and the fact that $\alpha < 1$, we find

$$\|\nabla\mathsf{u}^C\|_{L^\infty(\Omega,\mathbb{C}^3)} \leq C\varepsilon^{-1}.$$

By gauge invariance, $|\mathsf{u}| = |\mathsf{u}^C|$, and therefore

$$\|\nabla|\mathsf{u}|\|_{L^\infty(\Omega,\mathbb{R}^3)} = \|\nabla|\mathsf{u}^C|\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq \|\nabla \mathsf{u}^C\|_{L^\infty(\Omega,\mathbb{C}^3)} \leq \frac{C}{\varepsilon}.$$

Moreover, since

$$|u| = \frac{|\mathsf{u}|}{\rho_\varepsilon},$$

Proposition 2.1 yields

$$\|\nabla|u|\|_{L^\infty(\Omega,\mathbb{R}^3)} = \left\|\nabla\frac{|\mathsf{u}|}{\rho_\varepsilon}\right\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C\left(\|\nabla|\mathsf{u}|\|_{L^\infty(\Omega,\mathbb{R}^3)} + \|\nabla\rho_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^3)}\right) \leq \frac{C}{\varepsilon}.$$

We can therefore apply Lemma 2.1. For every $x_0 \in \overline{\Omega}$,

$$(1-|u|(x_0))^5 \leq \frac{C}{\varepsilon^3}\int_{B_\varepsilon(x_0)\cap\Omega}(1-|u|^2)^2 \leq \frac{C}{\varepsilon^3}\int_\Omega (1-|u|^2)^2 \leq \frac{C}{\varepsilon}F_{\varepsilon,\Omega}(u,A) = o(1).$$

We have thus proved that

$$\|1-|u|\|_{L^\infty(\Omega)} = o(1).$$

Since $|\mathsf{u}| = \rho_\varepsilon|u|$, this also implies that

$$\||\mathsf{u}| - \rho_\varepsilon\|_{L^\infty(\Omega)} = o(1).$$

This proves item (b).

We can now use [Rom19a, Proposition A.1] to improve our vorticity estimate. Since $|u| \geq \frac{1}{2}$ for all sufficiently small ε, we have

$$\|\mu(u,A)\|_{C_T^{0,1}(\Omega,\mathbb{R}^3)^*} \leq C\varepsilon F_{\varepsilon,\Omega}(u,A). \tag{3.9}$$

By interpolation with (2.14), we find, for any $\beta \in (0,1]$,

$$\|\mu(u,A)\|_{C_T^{0,\beta}(\Omega,\mathbb{R}^3)^*} \leq C\varepsilon^\beta F_{\varepsilon,\Omega}(u,A).$$

This allows us to improve the estimate on $F_{\varepsilon,\Omega}(u,A)$. Arguing as above, with $\beta = \beta_0$, and recalling (3.4) and (3.7), we obtain

$$F_{\varepsilon,\rho_\varepsilon}(u,A) \leq Ch_{\mathrm{ex}}\varepsilon^{\beta_0}F_{\varepsilon,\Omega}(u,A) + Ch_{\mathrm{ex}}^2\varepsilon^\vartheta F_{\varepsilon,\Omega}(u,A)^{\frac{\vartheta}{2}}. \tag{3.10}$$

Since $\beta_0 > 3\alpha > \alpha$, the first term on the right-hand side can be absorbed for all sufficiently small ε. Since $\frac{\vartheta}{2} \leq \frac{1}{2}$, using once again

$$F_{\varepsilon,\Omega}(u,A) \leq b^{-2}F_{\varepsilon,\rho_\varepsilon}(u,A),$$

we deduce from (3.10) that

$$F_{\varepsilon,\Omega}(u,A)^{1-\frac{\vartheta}{2}} \leq C\varepsilon^{\vartheta-2\alpha}.$$

Therefore,

$$F_{\varepsilon,\Omega}(u,A) \leq C\varepsilon^{\frac{\vartheta-2\alpha}{1-\frac{\vartheta}{2}}}.$$

Let

$$\delta := \frac{\vartheta-2\alpha}{1-\frac{\vartheta}{2}} - 1 = \frac{\frac{3\vartheta}{2}-2\alpha-1}{1-\frac{\vartheta}{2}}.$$

The inequality

$$\frac{3\vartheta}{2} - 2\alpha > 1$$

proved above shows that $\delta > 0$. Hence,

$$F_{\varepsilon,\Omega}(u, A) \leq C\varepsilon^{1+\delta}.$$

This proves item (c).

On the other hand, by direct computation,

$$\begin{aligned}
\mu\left(\frac{\mathsf{u}}{\rho_\varepsilon}, \mathsf{A}\right) &= \mu\left(\frac{\mathsf{u}}{\rho_\varepsilon}, \mathsf{A} - h_{\mathrm{ex}}(A^0_\varepsilon - \nabla\phi^0_\varepsilon) + h_{\mathrm{ex}}(A^0_\varepsilon - \nabla\phi^0_\varepsilon)\right) \\
&= \mu\left(\frac{\mathsf{u}}{\rho_\varepsilon}, \mathsf{A} - h_{\mathrm{ex}}(A^0_\varepsilon - \nabla\phi^0_\varepsilon)\right) + h_{\mathrm{ex}} \operatorname{curl}\left(\left(1 - \left|\frac{\mathsf{u}}{\rho_\varepsilon}\right|^2\right)(A^0_\varepsilon - \nabla\phi^0_\varepsilon)\right) \\
&= \mu(u, A) + h_{\mathrm{ex}} \operatorname{curl}\left((1 - |u|^2)\frac{\operatorname{curl} B^0_\varepsilon}{\rho^2_\varepsilon}\right).
\end{aligned}$$

Here, in the last equality, we used that

$$\left(\rho_\varepsilon^{-1}\mathsf{u}, \mathsf{A} - h_{\mathrm{ex}}(A^0_\varepsilon - \nabla\phi^0_\varepsilon)\right)$$

is gauge equivalent to (u, A), together with (1.7).

For any $B \in C^{0,1}_T(\Omega, \mathbb{R}^3)$, integration by parts gives

$$\begin{aligned}
\left|\int_\Omega h_{\mathrm{ex}} \operatorname{curl}\left((1 - |u|^2)\frac{\operatorname{curl} B^0_\varepsilon}{\rho^2_\varepsilon}\right) \cdot B\right| &= \left|h_{\mathrm{ex}} \int_\Omega (1 - |u|^2)\frac{\operatorname{curl} B^0_\varepsilon}{\rho^2_\varepsilon} \cdot \operatorname{curl} B\right| \\
&\leq C h_{\mathrm{ex}} \|1 - |u|^2\|_{L^2(\Omega)} \|\operatorname{curl} B^0_\varepsilon\|_{L^2(\Omega,\mathbb{R}^3)} \|B\|_{C^{0,1}_T(\Omega,\mathbb{R}^3)} \\
&\leq C h_{\mathrm{ex}} \varepsilon F_{\varepsilon,\Omega}(u, A)^{\frac{1}{2}} \|B\|_{C^{0,1}_T(\Omega,\mathbb{R}^3)} \\
&= o(1)\|B\|_{C^{0,1}_T(\Omega,\mathbb{R}^3)}.
\end{aligned}$$

We therefore deduce that

$$\left\|\mu\left(\frac{\mathsf{u}}{\rho_\varepsilon}, \mathsf{A}\right) - \mu(u, A)\right\|_{C^{0,1}_T(\Omega,\mathbb{R}^3)^*} = o(1).$$

Together with (3.9), this proves item (d).

Step 4. Convergence in the global divergence-free gauge.

By gauge invariance and Lemma A, we may assume that

$$\mathsf{A} - A_{\mathrm{ex}} \in \dot{H}^1_{\mathrm{div}=0}.$$

Recalling that

$$A = \mathsf{A} - h_{\mathrm{ex}} A^0_\varepsilon = (\mathsf{A} - A_{\mathrm{ex}}) - h_{\mathrm{ex}}(A^0_\varepsilon - A_{0,\mathrm{ex}}),$$

and using (1.6), we deduce that

$$A \in \dot{H}^1_{\mathrm{div}=0}.$$

Moreover,

$$\|A\|^2_{\dot{H}^1_{\mathrm{div}=0}} = \|\operatorname{curl} A\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \leq 2F_{\varepsilon,\rho_\varepsilon}(u, A) = o(\varepsilon). \tag{3.11}$$

On the other hand,

$$\begin{aligned}\int_\Omega |\nabla u|^2 &\le 2\int_\Omega |\nabla_A u|^2 + 2\int_\Omega |u|^2|A|^2\\ &\le C\int_\Omega \rho_\varepsilon^2|\nabla_A u|^2 + C\|u\|^2_{L^\infty(\Omega,\mathbb{C})}\|A\|^2_{L^2(\Omega,\mathbb{R}^3)}\\ &\le CF_{\varepsilon,\rho_\varepsilon}(u,A) + C\|A\|^2_{\dot H^1_{\mathrm{div}=0}}\\ &\le CF_{\varepsilon,\rho_\varepsilon}(u,A) = o(\varepsilon),\end{aligned}$$

where we used $b \le \rho_\varepsilon^2$, $\||1-|u|\|_{L^\infty(\Omega)} = o(1)$, and

$$\|A\|_{L^2(\Omega,\mathbb{R}^3)} \le C\|A\|_{L^6(\mathbb{R}^3,\mathbb{R}^3)} \le C\|A\|_{\dot H^1_{\mathrm{div}=0}}.$$

Letting

$$\underline{u} = \frac{1}{|\Omega|}\int_\Omega u,$$

the Poincaré–Wirtinger inequality yields

$$\int_\Omega |u-\underline{u}|^2 \le C\int_\Omega |\nabla u|^2 = o(\varepsilon). \tag{3.12}$$

Notice also that

$$\int_\Omega (1-|u|)^2 \le \int_\Omega (1-|u|^2)^2$$

and

$$\int_\Omega \big||u|-|\underline{u}|\big|^2 \le \int_\Omega |u-\underline{u}|^2.$$

We then deduce that

$$\int_\Omega |1-|\underline{u}||^2 \le C\int_\Omega (1-|u|^2)^2 + C\int_\Omega |u-\underline{u}|^2 = o(\varepsilon).$$

Since $\underline{u}$ is constant, it follows that

$$\underline{u} = (1+o(\sqrt{\varepsilon}))e^{i\theta} = e^{i\theta} + o(\sqrt{\varepsilon})$$

for some $\theta = \theta(\varepsilon) \in [0,2\pi]$. From (3.12), we obtain

$$\inf_{\theta\in[0,2\pi]} \|u - e^{i\theta}\|_{H^1(\Omega,\mathbb{C})} = o(\sqrt{\varepsilon}). \tag{3.13}$$

Combining this with (3.11), we have proved item (e).

Step 5. Convergence to the Meissner configuration.

Recalling that

$$\rho_\varepsilon^{-1}\mathsf{u} = ue^{ih_{\mathrm{ex}}\phi_\varepsilon^0},$$

we have

$$\inf_{\theta\in[0,2\pi]} \|\rho_\varepsilon^{-1}\mathsf{u} - e^{i\theta}e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}\|_{H^1(\Omega,\mathbb{C})} = \inf_{\theta\in[0,2\pi]} \|(u-e^{i\theta})e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}\|_{H^1(\Omega,\mathbb{C})}. \tag{3.14}$$

On the other hand, for any $\theta \in [0,2\pi]$,

$$\|(u-e^{i\theta})e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}\|_{H^1(\Omega,\mathbb{C})} \le C\left(\|u-e^{i\theta}\|_{L^2(\Omega,\mathbb{C})} + \|\nabla u\|_{L^2(\Omega,\mathbb{C}^3)} + h_{\mathrm{ex}}\|(u-e^{i\theta})\nabla\phi_\varepsilon^0\|_{L^2(\Omega,\mathbb{C}^3)}\right). \tag{3.15}$$

Recalling (1.10), since $p_0 > 3$, Hölder's inequality and Sobolev embedding yield

$$\|(u - e^{i\theta})\nabla\phi_\varepsilon^0\|_{L^2(\Omega,\mathbb{C}^3)} \leq \|u - e^{i\theta}\|_{L^{\frac{2p_0}{p_0-2}}(\Omega,\mathbb{C})}\|\nabla\phi_\varepsilon^0\|_{L^{p_0}(\Omega,\mathbb{R}^3)} \leq C\|u - e^{i\theta}\|_{H^1(\Omega,\mathbb{C})}. \tag{3.16}$$

Since $h_{\mathrm{ex}} \leq \varepsilon^{-\alpha}$ and $\alpha < \frac{1}{4}$, (3.13) and (3.16) imply that

$$h_{\mathrm{ex}} \inf_{\theta\in[0,2\pi]} \|(u - e^{i\theta})\nabla\phi_\varepsilon^0\|_{L^2(\Omega,\mathbb{C}^3)} = o(1).$$

Combining this with (3.14) and (3.15), we finally obtain

$$\inf_{\theta\in[0,2\pi]} \|\rho_\varepsilon^{-1}\mathsf{u} - e^{i\theta}e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}\|_{H^1(\Omega,\mathbb{C})} = o(1).$$

Since

$$\mathsf{A} - h_{\mathrm{ex}}A_\varepsilon^0 = A,$$

and $\|A\|_{\dot{H}^1_{\mathrm{div}=0}} = o(1)$, item (f) follows.

The theorem is thus proved. □

Remark 3.1. *The condition $p_0 > \frac{3}{1-3\alpha}$ arises from the preliminary estimate of the vorticity term. Indeed, Proposition 2.5 provides a gain of order $\varepsilon^{\frac{\beta}{3}}$ in the dual of $C_T^{0,\beta}(\Omega,\mathbb{R}^3)$, while*

$$W^{1,p_0}(\Omega) \hookrightarrow C^{0,1-\frac{3}{p_0}}(\overline{\Omega}).$$

Taking $\beta = 1 - \frac{3}{p_0}$ gives the condition

$$\frac{1}{3}\left(1 - \frac{3}{p_0}\right) > \alpha.$$

The natural gain in the vorticity estimate would be of order ε^β. This would yield the condition

$$p_0 > \frac{3}{1-\alpha},$$

which coincides with the condition appearing in the uniqueness result and seems to be the natural, and possibly sharp, threshold. Obtaining such a gain, however, appears to be very difficult within the available three-dimensional ε-level vortex construction.

4. Uniqueness

We now present the proof of Theorem 1.2.

Proof. We proceed in several steps.

Step 1. Coulomb gauge estimates and clearing out.

Let us assume towards a contradiction that there exist two local minimizers $(\mathsf{u}_j, \mathsf{A}_j)$ satisfying the hypotheses of the theorem which are not gauge equivalent. Writing

$$(\mathsf{u}_j, \mathsf{A}_j) = (\rho_\varepsilon u_j e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, A_j + h_{\mathrm{ex}}A_\varepsilon^0),$$

we see that this configuration is gauge equivalent to

$$(\rho_\varepsilon u_j, A_j + h_{\mathrm{ex}}(A_\varepsilon^0 - \nabla\phi_\varepsilon^0)).$$

We gauge transform the latter configuration into $(\mathsf{u}_j^C, \mathsf{A}_j^C)$, where A_j^C is in the Coulomb gauge in Ω. Thus,

$$\begin{cases} \operatorname{div} \mathsf{A}_j^C &= 0 \quad \text{in } \Omega \\ \mathsf{A}_j^C \cdot \nu &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

In particular,

$$|\mathsf{u}_j^C| = \rho_\varepsilon |u_j|.$$

Using the gauge invariance of GL_ε, we deduce that $(\mathsf{u}_j^C, \mathsf{A}_j^C)$ is a critical point. Proposition 2.3 then yields

$$\|\mathsf{u}_j^C\|_{L^\infty(\Omega,\mathbb{C})} \leq 1 \quad \text{and} \quad \left\| \frac{\mathsf{u}_j^C}{\rho_\varepsilon} \right\|_{L^\infty(\Omega,\mathbb{C})} \leq 1.$$

Since $(\mathsf{u}_j^C, \mathsf{A}_j^C)$ is in the Coulomb gauge in Ω, Proposition 2.4 gives

$$\|\mathsf{A}_j^C\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C \left(\left\| \nabla_{\mathsf{A}_j^C} \left(\frac{\mathsf{u}_j^C}{\rho_\varepsilon} \right) \right\|_{L^2(\Omega,\mathbb{C}^3)} + h_{\text{ex}} \right).$$

Moreover,

$$\left(\frac{\mathsf{u}_j^C}{\rho_\varepsilon}, \mathsf{A}_j^C \right)$$

is gauge equivalent to

$$(u_j, A_j + h_{\text{ex}}(A_\varepsilon^0 - \nabla \phi_\varepsilon^0)).$$

Therefore, by the gauge invariance of the covariant gradient,

$$\begin{aligned} \left\| \nabla_{\mathsf{A}_j^C} \left(\frac{\mathsf{u}_j^C}{\rho_\varepsilon} \right) \right\|_{L^2(\Omega,\mathbb{C}^3)}^2 &= \|\nabla_{A_j + h_{\text{ex}}(A_\varepsilon^0 - \nabla\phi_\varepsilon^0)} u_j\|_{L^2(\Omega,\mathbb{C}^3)}^2 \\ &\leq C \left(\|\nabla_{A_j} u_j\|_{L^2(\Omega,\mathbb{C}^3)}^2 + h_{\text{ex}}^2 \|u_j (A_\varepsilon^0 - \nabla\phi_\varepsilon^0)\|_{L^2(\Omega,\mathbb{C}^3)}^2 \right) \\ &\leq C \left(F_{\varepsilon,\Omega}(u_j, A_j) + h_{\text{ex}}^2 \right). \end{aligned}$$

Here, we used (1.7), together with $\|u_j\|_{L^\infty(\Omega,\mathbb{C})} \leq 1$ and $\|\operatorname{curl} B_\varepsilon^0\|_{L^2(\Omega,\mathbb{R}^3)} \leq C$. Since

$$F_{\varepsilon,\Omega}(u_j, A_j) \leq c_0 \varepsilon$$

and $h_{\text{ex}} \leq \varepsilon^{-\alpha}$, we conclude that

$$\|\mathsf{A}_j^C\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C\varepsilon^{-\alpha}. \tag{4.1}$$

In turn, Proposition 2.3 and the fact that $\alpha < 1$ yield

$$\|\nabla \mathsf{u}_j^C\|_{L^\infty(\Omega,\mathbb{C}^3)} \leq C \left(\varepsilon^{-1} + \|\mathsf{A}_j^C\|_{L^\infty(\Omega,\mathbb{R}^3)} \right) \leq C\varepsilon^{-1}.$$

Using

$$|\nabla |\mathsf{u}_j^C|| \leq |\nabla \mathsf{u}_j^C|$$

and

$$|u_j| = \rho_\varepsilon^{-1} |\mathsf{u}_j^C|,$$

together with Proposition 2.1, we find

$$\|\nabla |u_j|\|_{L^\infty(\Omega,\mathbb{R}^3)} = \|\nabla(\rho_\varepsilon^{-1} |\mathsf{u}_j^C|)\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C\varepsilon^{-1}.$$

We can now apply Lemma 2.1. Denoting by $C_{\rm co} > 0$ the constant in the clearing-out estimate, for every $x_0 \in \overline{\Omega}$ we have

$$(1 - |u_j|(x_0))^5 \leq \frac{C_{\rm co}}{\varepsilon^3} \int_{B_\varepsilon(x_0)\cap\Omega} (1 - |u_j|^2)^2 \leq \frac{C_{\rm co}}{\varepsilon} F_{\varepsilon,\Omega}(u_j, A_j) \leq C_{\rm co} c_0.$$

Let

$$\tau := (C_{\rm co} c_0)^{\frac{1}{5}}.$$

We choose $c_0 > 0$ sufficiently small so that

$$\tau < \frac{1}{4} \quad \text{and} \quad 4\tau \leq \frac{\delta_0}{2}, \tag{4.2}$$

where $\delta_0 = \delta_0(b, p_0, r_0)$ is the constant in (1.11). In particular, c_0 depends on the fixed parameters of the problem and on p_0.

By the definition of τ, we have

$$\frac{3}{4} \leq 1 - \tau \leq |u_j| \leq 1 \quad \text{in } \Omega.$$

We may thus write

$$\frac{\mathsf{u}_j^C}{\rho_\varepsilon} = \eta_j e^{i\phi_j},$$

where

$$\eta_j = \left|\frac{\mathsf{u}_j^C}{\rho_\varepsilon}\right| = |u_j|, \quad \text{with } \frac{3}{4} \leq 1 - \tau \leq \eta_j \leq 1. \tag{4.3}$$

Step 2. Elimination of the phase.

We gauge transform $(\mathsf{u}_j^C, \mathsf{A}_j^C)$ into $(\widetilde{\mathsf{u}}_j, \widetilde{\mathsf{A}}_j)$, where

$$\widetilde{\mathsf{u}}_j = \rho_\varepsilon \eta_j, \qquad \widetilde{\mathsf{A}}_j = \mathsf{A}_j^C - \nabla\phi_j.$$

After extending ϕ_j to a function in $H^1_{\rm loc}(\mathbb{R}^3)$, this gauge transformation is defined in the whole space. In particular, $(\widetilde{\mathsf{u}}_j, \widetilde{\mathsf{A}}_j)$ is a critical point of GL_ε and is gauge equivalent to $(\mathsf{u}_j, \mathsf{A}_j)$.

Since $(\widetilde{\mathsf{u}}_j, \widetilde{\mathsf{A}}_j)$ satisfies the second equation in (2.5), we have

$$\operatorname{curl}^2(\widetilde{\mathsf{A}}_j - A_{\rm ex}) = (i\widetilde{\mathsf{u}}_j, \nabla_{\widetilde{\mathsf{A}}_j}\widetilde{\mathsf{u}}_j)\chi_\Omega \quad \text{in } \mathbb{R}^3.$$

Taking the divergence of this equation in Ω, we find

$$\operatorname{div}\left((i\widetilde{\mathsf{u}}_j, \nabla_{\widetilde{\mathsf{A}}_j}\widetilde{\mathsf{u}}_j)\right) = 0 \quad \text{in } \Omega.$$

Since $\widetilde{\mathsf{u}}_j = \rho_\varepsilon\eta_j$ is real-valued,

$$(i\widetilde{\mathsf{u}}_j, \nabla_{\widetilde{\mathsf{A}}_j}\widetilde{\mathsf{u}}_j) = -\rho_\varepsilon^2\eta_j^2\widetilde{\mathsf{A}}_j \quad \text{in } \Omega.$$

Hence,

$$\operatorname{div}(\eta_j^2\rho_\varepsilon^2\widetilde{\mathsf{A}}_j) = 0 \quad \text{in } \Omega.$$

The first boundary condition in (2.5) also yields

$$\widetilde{\mathsf{A}}_j \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

because $\widetilde{\mathsf{u}}_j$ is real-valued and strictly positive. Recalling that

$$\widetilde{\mathsf{A}}_j = \mathsf{A}_j^C - \nabla\phi_j,$$

we conclude that

$$\begin{cases} \operatorname{div}\left(\eta_j^2\rho_\varepsilon^2(\mathsf{A}_j^C - \nabla\phi_j)\right) &= 0 \quad \text{in } \Omega \\ (\mathsf{A}_j^C - \nabla\phi_j)\cdot\nu &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Thus, $\mathsf{A}_j^C - \nabla\phi_j$ is in the weighted Coulomb gauge associated with the weight $\eta_j^2\rho_\varepsilon^2$. This is the only point in the paper where we need to apply the weighted elliptic estimate to a perturbation of the original weight ρ_ε^2. The factor $\frac{1}{2}$ in (1.11) was introduced precisely to ensure that the small-BMO condition required in [BW05, Theorem 2] remains valid after this perturbation.

Indeed, by assumption (1.11),

$$[\rho_\varepsilon^2]_{BMO_{r_0}(\Omega)} \le \frac{\delta_0}{2}.$$

Moreover, (4.3) implies that

$$\|\eta_j^2 - 1\|_{L^\infty(\Omega)} \le 2\tau.$$

For any $f \in L^\infty(\Omega)$, we have

$$[f]_{BMO_{r_0}(\Omega)} \le 2\|f\|_{L^\infty(\Omega)}.$$

Therefore, using $\rho_\varepsilon^2 \le 1$, we obtain

$$[(\eta_j^2 - 1)\rho_\varepsilon^2]_{BMO_{r_0}(\Omega)} \le 2\|(\eta_j^2 - 1)\rho_\varepsilon^2\|_{L^\infty(\Omega)} \le 2\|\eta_j^2 - 1\|_{L^\infty(\Omega)}.$$

Consequently,

$$\begin{aligned}[\eta_j^2\rho_\varepsilon^2]_{BMO_{r_0}(\Omega)} \le [\rho_\varepsilon^2]_{BMO_{r_0}(\Omega)} + [(\eta_j^2 - 1)\rho_\varepsilon^2]_{BMO_{r_0}(\Omega)} &\le \frac{\delta_0}{2} + 2\|\eta_j^2 - 1\|_{L^\infty(\Omega)} \\ &\le \frac{\delta_0}{2} + 4\tau \le \delta_0,\end{aligned}$$

where in the last inequality we used (4.2).

Therefore, [BW05, Theorem 2] applied to (4.4) yields

$$\|\nabla\phi_j\|_{L^{p_0}(\Omega,\mathbb{R}^3)} \le C\|\mathsf{A}_j^C\|_{L^{p_0}(\Omega,\mathbb{R}^3)}.$$

Combining this with (4.1), we find

$$\|\nabla\phi_j\|_{L^{p_0}(\Omega,\mathbb{R}^3)} \le C\varepsilon^{-\alpha}.$$

It follows that

$$\|\widetilde{\mathsf{A}}_j\|_{L^{p_0}(\Omega,\mathbb{R}^3)} \le \|\mathsf{A}_j^C\|_{L^{p_0}(\Omega,\mathbb{R}^3)} + \|\nabla\phi_j\|_{L^{p_0}(\Omega,\mathbb{R}^3)} \le C\varepsilon^{-\alpha}. \tag{4.5}$$

Step 3. Strict energy comparison.

We prove that, for all sufficiently small ε,

$$\frac{Y}{2} := \frac{GL_\varepsilon(\widetilde{\mathsf{u}}_1, \widetilde{\mathsf{A}}_1) + GL_\varepsilon(\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2)}{2} - GL_\varepsilon\left(\frac{\widetilde{\mathsf{u}}_1 + \widetilde{\mathsf{u}}_2}{2}, \frac{\widetilde{\mathsf{A}}_1 + \widetilde{\mathsf{A}}_2}{2}\right) > 0. \tag{4.6}$$

Recalling that $\widetilde{\mathsf{u}}_j = \rho_\varepsilon \eta_j$ and using (1.4), we find

$$GL_\varepsilon(\widetilde{\mathsf{u}}_j, \widetilde{\mathsf{A}}_j) = E_\varepsilon(\rho_\varepsilon) + \frac{1}{2}\int_\Omega \left(\rho_\varepsilon^2 |\nabla \eta_j|^2 + \rho_\varepsilon^2 \eta_j^2 |\widetilde{\mathsf{A}}_j|^2 + \frac{\rho_\varepsilon^4}{\varepsilon^2}(1-\eta_j^2)^2\right) + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl} \widetilde{\mathsf{A}}_j - H_{\mathrm{ex}}|^2.$$

We write

$$Y = Y_0 + Y_1 + Y_2 + Y_3,$$

where

$$Y_0 = \frac{1}{2}\int_\Omega \rho_\varepsilon^2 \left(|\nabla \eta_1|^2 + |\nabla \eta_2|^2\right) - \int_\Omega \rho_\varepsilon^2 \left|\nabla\left(\frac{\eta_1+\eta_2}{2}\right)\right|^2$$

$$Y_1 = \frac{1}{2}\int_\Omega \rho_\varepsilon^2 \left(\eta_1^2 |\widetilde{\mathsf{A}}_1|^2 + \eta_2^2 |\widetilde{\mathsf{A}}_2|^2\right) - \int_\Omega \rho_\varepsilon^2 \left(\frac{\eta_1+\eta_2}{2}\right)^2 \left|\frac{\widetilde{\mathsf{A}}_1 + \widetilde{\mathsf{A}}_2}{2}\right|^2$$

$$Y_2 = \frac{1}{4\varepsilon^2}\int_\Omega \rho_\varepsilon^4 \left((1-\eta_1^2)^2 + (1-\eta_2^2)^2\right) - \frac{1}{2\varepsilon^2}\int_\Omega \rho_\varepsilon^4 \left(1 - \left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2$$

$$Y_3 = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\operatorname{curl} \widetilde{\mathsf{A}}_1 - H_{\mathrm{ex}}|^2 + |\operatorname{curl} \widetilde{\mathsf{A}}_2 - H_{\mathrm{ex}}|^2\right) - \int_{\mathbb{R}^3} \left|\operatorname{curl}\left(\frac{\widetilde{\mathsf{A}}_1 + \widetilde{\mathsf{A}}_2}{2}\right) - H_{\mathrm{ex}}\right|^2.$$

Observe that $Y_3 \geq 0$ by convexity. We next prove that $Y_0 + Y_1 + Y_2 > 0$. A direct computation gives

$$Y_0 = \frac{1}{4}\int_\Omega \rho_\varepsilon^2 |\nabla(\eta_1 - \eta_2)|^2 \geq \frac{b}{4}\|\nabla(\eta_1 - \eta_2)\|^2_{L^2(\Omega,\mathbb{R}^3)}. \tag{4.7}$$

Moreover,

$$Y_1 = \int_\Omega \rho_\varepsilon^2 \Bigg(\frac{1}{8}(\eta_1-\eta_2)^2\left(|\widetilde{\mathsf{A}}_1|^2 + |\widetilde{\mathsf{A}}_2|^2\right) + \frac{1}{16}(3\eta_1^2 - 2\eta_1\eta_2 + 3\eta_2^2)|\widetilde{\mathsf{A}}_1 - \widetilde{\mathsf{A}}_2|^2 + \frac{1}{8}(\eta_1-\eta_2)(\widetilde{\mathsf{A}}_1 - \widetilde{\mathsf{A}}_2)\cdot\left((\eta_1+3\eta_2)\widetilde{\mathsf{A}}_1 + (3\eta_1+\eta_2)\widetilde{\mathsf{A}}_2\right)\Bigg). \tag{4.8}$$

Finally,

$$Y_2 = \frac{1}{32\varepsilon^2}\int_\Omega \rho_\varepsilon^4 (\eta_1-\eta_2)^2\left(7(\eta_1+\eta_2)^2 - 4\eta_1\eta_2 - 8\right).$$

In view of (4.3), we have

$$7(\eta_1+\eta_2)^2 - 4\eta_1\eta_2 - 8 \geq \frac{11}{2}.$$

Using also $\rho_\varepsilon^4 \geq b^2$, we conclude that

$$Y_2 \geq \frac{11 b^2}{64\varepsilon^2}\|\eta_1 - \eta_2\|^2_{L^2(\Omega)}. \tag{4.9}$$

We now estimate the last term in (4.8). Since $\rho_\varepsilon^2 \leq 1$ and $\eta_j \leq 1$, its absolute value is bounded by

$$C\int_\Omega |\eta_1 - \eta_2||\widetilde{\mathsf{A}}_1 - \widetilde{\mathsf{A}}_2|\left(|\widetilde{\mathsf{A}}_1| + |\widetilde{\mathsf{A}}_2|\right).$$

Let

$$r = \frac{2p_0}{p_0 - 2},$$

so that

$$\frac{1}{r}+\frac{1}{2}+\frac{1}{p_0}=1.$$

Therefore, by Hölder's inequality and (4.5),

$$\begin{aligned}\int_\Omega |\eta_1-\eta_2||\widetilde{\mathsf{A}}_1-\widetilde{\mathsf{A}}_2|\left(|\widetilde{\mathsf{A}}_1|+|\widetilde{\mathsf{A}}_2|\right)&\\ \le \|\eta_1-\eta_2\|_{L^r(\Omega)}\|\widetilde{\mathsf{A}}_1-\widetilde{\mathsf{A}}_2\|_{L^2(\Omega,\mathbb{R}^3)}\left(\|\widetilde{\mathsf{A}}_1\|_{L^{p_0}(\Omega,\mathbb{R}^3)}+\|\widetilde{\mathsf{A}}_2\|_{L^{p_0}(\Omega,\mathbb{R}^3)}\right)&\\ \le C\varepsilon^{-\alpha}\|\eta_1-\eta_2\|_{L^r(\Omega)}\|\widetilde{\mathsf{A}}_1-\widetilde{\mathsf{A}}_2\|_{L^2(\Omega,\mathbb{R}^3)}.&\end{aligned}$$

Since $p_0>3$, we have $2<r<6$. Interpolation and Sobolev embedding give

$$\begin{aligned}\|\eta_1-\eta_2\|_{L^r(\Omega)}&\le C\|\eta_1-\eta_2\|_{L^2(\Omega)}^{1-\frac{3}{p_0}}\|\eta_1-\eta_2\|_{L^6(\Omega)}^{\frac{3}{p_0}}\\ &\le C\|\eta_1-\eta_2\|_{L^2(\Omega)}^{1-\frac{3}{p_0}}\left(\|\nabla(\eta_1-\eta_2)\|_{L^2(\Omega,\mathbb{R}^3)}+\|\eta_1-\eta_2\|_{L^2(\Omega)}\right)^{\frac{3}{p_0}}.\end{aligned}$$

Writing

$$\|\eta_1-\eta_2\|_{L^2(\Omega)}=\varepsilon\left(\varepsilon^{-2}\|\eta_1-\eta_2\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}$$

and observing that, for $\varepsilon<1$,

$$\begin{aligned}\|\nabla(\eta_1-\eta_2)\|_{L^2(\Omega,\mathbb{R}^3)}+\|\eta_1-\eta_2\|_{L^2(\Omega)}&\\ \le C\left(\|\nabla(\eta_1-\eta_2)\|_{L^2(\Omega,\mathbb{R}^3)}^2+\varepsilon^{-2}\|\eta_1-\eta_2\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}},&\end{aligned}$$

it follows that

$$\|\eta_1-\eta_2\|_{L^r(\Omega)}\le C\varepsilon^{1-\frac{3}{p_0}}\left(\|\nabla(\eta_1-\eta_2)\|_{L^2(\Omega,\mathbb{R}^3)}^2+\varepsilon^{-2}\|\eta_1-\eta_2\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}.$$

Consequently,

$$\begin{aligned}\int_\Omega |\eta_1-\eta_2||\widetilde{\mathsf{A}}_1-\widetilde{\mathsf{A}}_2|\left(|\widetilde{\mathsf{A}}_1|+|\widetilde{\mathsf{A}}_2|\right)&\\ \le C\varepsilon^{1-\alpha-\frac{3}{p_0}}\left(\|\nabla(\eta_1-\eta_2)\|_{L^2(\Omega,\mathbb{R}^3)}^2+\varepsilon^{-2}\|\eta_1-\eta_2\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}\|\widetilde{\mathsf{A}}_1-\widetilde{\mathsf{A}}_2\|_{L^2(\Omega,\mathbb{R}^3)}.&\end{aligned}$$

By Young's inequality,

$$\begin{aligned}\int_\Omega |\eta_1-\eta_2||\widetilde{\mathsf{A}}_1-\widetilde{\mathsf{A}}_2|\left(|\widetilde{\mathsf{A}}_1|+|\widetilde{\mathsf{A}}_2|\right)&\\ \le C\varepsilon^{1-\alpha-\frac{3}{p_0}}\left(\|\nabla(\eta_1-\eta_2)\|_{L^2(\Omega,\mathbb{R}^3)}^2+\varepsilon^{-2}\|\eta_1-\eta_2\|_{L^2(\Omega)}^2+\|\widetilde{\mathsf{A}}_1-\widetilde{\mathsf{A}}_2\|_{L^2(\Omega,\mathbb{R}^3)}^2\right).&\end{aligned}\tag{4.10}$$

Since

$$p_0>\frac{3}{1-\alpha},$$

we have

$$\varepsilon^{1-\alpha-\frac{3}{p_0}}=o(1).$$

In view of (4.3), we have

$$3\eta_1^2-2\eta_1\eta_2+3\eta_2^2=(\eta_1-\eta_2)^2+2\eta_1^2+2\eta_2^2\ge\frac{9}{4}.$$

Therefore, the second term in (4.8) satisfies

$$\frac{1}{16}\int_\Omega \rho_\varepsilon^2(3\eta_1^2 - 2\eta_1\eta_2 + 3\eta_2^2)|\widetilde{\mathsf{A}}_1 - \widetilde{\mathsf{A}}_2|^2 \geq \frac{9b}{64}\|\widetilde{\mathsf{A}}_1 - \widetilde{\mathsf{A}}_2\|^2_{L^2(\Omega,\mathbb{R}^3)}. \tag{4.11}$$

Combining (4.7), (4.8), (4.9), (4.10), and (4.11), and discarding the first nonnegative term in (4.8), we obtain

$$\begin{aligned} Y_0 + Y_1 + Y_2 \geq &\left(\frac{b}{4} - o(1)\right)\|\nabla(\eta_1 - \eta_2)\|^2_{L^2(\Omega,\mathbb{R}^3)} \\ &+ \left(\frac{11b^2}{64} - o(1)\right)\varepsilon^{-2}\|\eta_1 - \eta_2\|^2_{L^2(\Omega)} + \left(\frac{9b}{64} - o(1)\right)\|\widetilde{\mathsf{A}}_1 - \widetilde{\mathsf{A}}_2\|^2_{L^2(\Omega,\mathbb{R}^3)}. \end{aligned}$$

Therefore,

$$Y_0 + Y_1 + Y_2 > 0$$

for all sufficiently small ε, unless

$$\eta_1 = \eta_2 \quad \text{and} \quad \widetilde{\mathsf{A}}_1 = \widetilde{\mathsf{A}}_2.$$

Since the two configurations are not gauge equivalent, the latter alternative is impossible. Recalling that $Y_3 \geq 0$, we conclude that

$$Y > 0.$$

Step 4. Contradiction.

We may assume without loss of generality that

$$GL_\varepsilon(\widetilde{\mathsf{u}}_1, \widetilde{\mathsf{A}}_1) \leq GL_\varepsilon(\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2).$$

Let

$$(\mathsf{u}^{(1)}, \mathsf{A}^{(1)}) := \left(\frac{\widetilde{\mathsf{u}}_1 + \widetilde{\mathsf{u}}_2}{2}, \frac{\widetilde{\mathsf{A}}_1 + \widetilde{\mathsf{A}}_2}{2}\right).$$

From (4.6), we have

$$GL_\varepsilon(\mathsf{u}^{(1)}, \mathsf{A}^{(1)}) < \frac{GL_\varepsilon(\widetilde{\mathsf{u}}_1, \widetilde{\mathsf{A}}_1) + GL_\varepsilon(\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2)}{2} \leq GL_\varepsilon(\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2).$$

We define inductively

$$(\mathsf{u}^{(n+1)}, \mathsf{A}^{(n+1)}) := \left(\frac{\mathsf{u}^{(n)} + \widetilde{\mathsf{u}}_2}{2}, \frac{\mathsf{A}^{(n)} + \widetilde{\mathsf{A}}_2}{2}\right).$$

Each configuration $(\mathsf{u}^{(n)}, \mathsf{A}^{(n)})$ is of the form

$$(\mathsf{u}^{(n)}, \mathsf{A}^{(n)}) = (\rho_\varepsilon \eta^{(n)}, \mathsf{A}^{(n)}),$$

where, by construction and in view of (4.3),

$$\frac{3}{4} \leq 1 - \tau \leq \eta^{(n)} \leq 1.$$

Moreover, (4.5) yields

$$\|\mathsf{A}^{(n)}\|_{L^{p_0}(\Omega,\mathbb{R}^3)} \leq C\varepsilon^{-\alpha}$$

uniformly in n. The argument of the previous step therefore applies to the pair $(\mathsf{u}^{(n)}, \mathsf{A}^{(n)})$ and $(\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2)$ and gives

$$GL_\varepsilon(\mathsf{u}^{(n+1)}, \mathsf{A}^{(n+1)}) < \frac{GL_\varepsilon(\mathsf{u}^{(n)}, \mathsf{A}^{(n)}) + GL_\varepsilon(\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2)}{2} < GL_\varepsilon(\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2).$$

It follows by induction that

$$GL_\varepsilon(\mathsf{u}^{(n)}, \mathsf{A}^{(n)}) < GL_\varepsilon(\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2)$$

for every $n \geq 1$.

On the other hand,

$$(\mathsf{u}^{(n)}, \mathsf{A}^{(n)}) - (\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2) = 2^{-n} \left((\widetilde{\mathsf{u}}_1, \widetilde{\mathsf{A}}_1) - (\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2) \right).$$

Therefore,

$$(\mathsf{u}^{(n)}, \mathsf{A}^{(n)}) \longrightarrow (\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2)$$

with respect to the natural topology of $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$. This contradicts the local minimality of $(\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2)$. Hence,

$$(\widetilde{\mathsf{u}}_1, \widetilde{\mathsf{A}}_1) = (\widetilde{\mathsf{u}}_2, \widetilde{\mathsf{A}}_2).$$

The theorem is thus proved. □

References

[AS13] C. Amrouche and N. E. H. Seloula, *L^p-theory for vector potentials and Sobolev's inequalities for vector fields: application to the Stokes equations with pressure boundary conditions*, Math. Models Methods Appl. Sci. **23** (2013), no. 1, 37–92. MR2997467 ↑7, 11

[BBH93] F. Bethuel, H. Brezis, and F. Hélein, *Asymptotics for the minimization of a Ginzburg-Landau functional*, Calc. Var. Partial Differential Equations **1** (1993), no. 2, 123–148. MR1261720 ↑12

[BBH94] F. Bethuel, H. Brezis, and F. Hélein, *Ginzburg-Landau vortices*, Progress in Nonlinear Differential Equations and their Applications, vol. 13, Birkhäuser Boston, Inc., Boston, MA, 1994. MR1269538 ↑3

[BBO01] F. Bethuel, H. Brezis, and G. Orlandi, *Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions*, J. Funct. Anal. **186** (2001), no. 2, 432–520. MR1864830 ↑16

[BCS57] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Theory of superconductivity*, Phys. Rev. **108** (1957), no. 5, 1175–1204. ↑3

[BJOS12] S. Baldo, R. L. Jerrard, G. Orlandi, and H. M. Soner, *Convergence of Ginzburg-Landau functionals in three-dimensional superconductivity*, Arch. Ration. Mech. Anal. **205** (2012), no. 3, 699–752. MR2960031 ↑4

[BJOS13] S. Baldo, R. L. Jerrard, G. Orlandi, and H. M. Soner, *Vortex density models for superconductivity and superfluidity*, Comm. Math. Phys. **318** (2013), no. 1, 131–171. MR3017066 ↑4

[BW05] S.-S. Byun and L. Wang, *The conormal derivative problem for elliptic equations with BMO coefficients on Reifenberg flat domains*, Proc. London Math. Soc. (3) **90** (2005), no. 1, 245–272. MR2107043 ↑7, 11, 29

[Cha95] S. J. Chapman, *Superheating field of type II superconductors*, SIAM J. Appl. Math. **55** (1995), no. 5, 1233–1258. MR1349308 ↑2

[CLC+24] C. Chen, Y. Liu, Y. Chen, Y. N. Hu, T. Z. Zhang, D. Li, X. Wang, C. X. Wang, Z. Y. W. Lu, Y. H. Zhang, Q. L. Zhang, X. L. Dong, R. Wang, D. L. Feng, and T. Zhang, *Revealing the microscopic mechanism of elementary vortex pinning in superconductors*, Phys. Rev. X **14** (2024), 041039. ↑2

[CR97] S. J. Chapman and G. Richardson, *Vortex pinning by inhomogeneities in type-II superconductors*, Phys. D **108** (1997), no. 4, 397–407. MR1474691 ↑2

[DS12] M. Dos Santos, *Study of a 3D-Ginzburg-Landau functional with a discontinuous pinning term*, Nonlinear Anal. **75** (2012), no. 17, 6275–6296. MR2959805 ↑4

[DSRS23] M. Dos Santos, R. Rodiac, and E. Sandier, *The Ginzburg-Landau energy with a pinning term oscillating faster than the coherence length*, Interfaces Free Bound. **25** (2023), no. 3, 491–515. MR4642021 ↑4

[DVR26a] M. Díaz-Vera and C. Román, *First critical field in the pinned three-dimensional Ginzburg-Landau model of superconductivity*, Calc. Var. Partial Differential Equations **65** (2026), no. 7, Paper No. 217, 37. MR5090628 ↑4, 6, 7, 15

[DVR26b] M. Díaz-Vera and C. Román, *On the Meissner state for type-II inhomogeneous superconductors*, J. Funct. Anal. **290** (2026), no. 7, Paper No. 111332, 45. MR5013119 ↑4

[DVRY] M. Díaz-Vera, C. Román, and K. Yamamoto. Work in progress. ↑7

[GL50] V. L. Ginzburg and L. D. Landau, *On the theory of superconductivity*, Zh. Eksp. Teor. Fiz. **20** (1950), 1064–1082. English translation in: *Collected papers of L.D.Landau*, Edited by D. Ter. Haar, Pergamon Press, Oxford 1965, pp. 546–568. ↑1

[Grö89] K. Gröger, *A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations*, Math. Ann. **283** (1989), no. 4, 679–687. MR990595 ↑7

[Jer99] R. L. Jerrard, *Lower bounds for generalized Ginzburg-Landau functionals*, SIAM J. Math. Anal. **30** (1999), no. 4, 721–746. MR1684723 ↑17

[KS91] H. Kozono and H. Sohr, *New a priori estimates for the Stokes equations in exterior domains*, Indiana Univ. Math. J. **40** (1991), no. 1, 1–27. MR1101219 ↑11

[KWG+16] W.-K. Kwok, U. Welp, A. Glatz, A. E. Koshelev, K. J. Kihlstrom, and G. W. Crabtree, *Vortices in high-performance high-temperature superconductors*, Rep. Prog. Phys. **79** (2016), no. 11, 116501, 39. ↑2

[Lik79] K. K. Likharev, *Superconducting weak links*, Reviews of Modern Physics **51** (1979), no. 1, 101–159. ↑2

[LM99] L. Lassoued and P. Mironescu, *Ginzburg-Landau type energy with discontinuous constraint*, J. Anal. Math. **77** (1999), 1–26. MR1753480 ↑5

[LR99] F. Lin and T. Rivière, *Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents*, J. Eur. Math. Soc. (JEMS) **1** (1999), no. 3, 237–311. MR1714735 ↑4

[Mey63] N. G. Meyers, *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **17** (1963), 189–206. MR159110 ↑7

[Riv95] T. Rivière, *Line vortices in the* U(1)*-Higgs model*, ESAIM Contrôle Optim. Calc. Var. **1** (1995/96), 77–167. MR1394302 ↑4

[Rom19a] C. Román, *On the first critical field in the three dimensional Ginzburg-Landau model of superconductivity*, Comm. Math. Phys. **367** (2019), no. 1, 317–349. MR3933412 ↑2, 4, 11, 13, 23

[Rom19b] C. Román, *Three dimensional vortex approximation construction and ε-level estimates for the Ginzburg-Landau functional*, Arch. Ration. Mech. Anal. **231** (2019), no. 3, 1531–1614. MR3902469 ↑4, 16, 17, 18

[Rom26] C. Román, *First critical field in the pinned three-dimensional Ginzburg–Landau model: A matching upper bound*, Ann. Henri Poincaré (2026). Available at https://doi.org/10.1007/s00023-026-01754-4. ↑4

[RSS23] C. Román, E. Sandier, and S. Serfaty, *Bounded vorticity for the 3D Ginzburg-Landau model and an isoflux problem*, Proc. Lond. Math. Soc. (3) **126** (2023), no. 3, 1015–1062. MR4563866 ↑4

[RSS25] C. Román, E. Sandier, and S. Serfaty, *Vortex lines interaction in the three-dimensional magnetic Ginzburg–Landau model*, 2025. Available at arXiv:2510.14910. ↑4

[San01] E. Sandier, *Ginzburg-Landau minimizers from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ and minimal connections*, Indiana Univ. Math. J. **50** (2001), no. 4, 1807–1844. MR1889083 ↑4

[Ser99] S. Serfaty, *Stable configurations in superconductivity: uniqueness, multiplicity, and vortex-nucleation*, Arch. Ration. Mech. Anal. **149** (1999), no. 4, 329–365. MR1731999 ↑3, 4

[SS07] E. Sandier and S. Serfaty, *Vortices in the magnetic Ginzburg-Landau model*, Progress in Nonlinear Differential Equations and their Applications, vol. 70, Birkhäuser Boston, Inc., Boston, MA, 2007. MR2279839 ↑3

[Tin96] M. Tinkham, *Introduction to superconductivity*, Second, McGraw-Hill, New York, 1996. ↑2